MATH 221 Solutions to the sample final

2 (a). True. $c_i\vec{v}_i \cdot c_j\vec{v}_j = 0$ iff $i \neq j$.

(b). True. This is the Pythagorean theorem. For an argument, note that $||\vec{v} - \vec{u}||^2 = ||\vec{v}||^2 + ||\vec{u}||^2 - 2\vec{u}\cdot\vec{v}$. Hence the assumption forces $\vec{u}\cdot\vec{v} = 0$ so that $\vec{u}$ and $\vec{v}$ are orthogonal.

(c). False. This is not true since the $\vec{0}$ vector is orthogonal to any vector. For an example, note that $\{(1,0),(0,0)\}$ is an orthogonal subset of $\mathbb{R}^2$. Otherwise this is true if we consider non-zero orthogonal vectors; this is a theorem from class.

(d). False. Let $A = \begin{bmatrix} 1 & 0 \end{bmatrix}$ so A is a 1×2 matrix. The Nullspace of A is spanned by $(0,1)$ while the column space is a subspace of $\mathbb{R}^1$. What is true is that the orthogonal complement of the row space of A is the nullspace of A.

(e). True. A theorem from class.

(f). True. If $\vec{x}$ is in $W^\perp$ then $\vec{x}$ is orthogonal to every vector in W and hence the $\vec{v}_j's$. If $\vec{x}$ is orthogonal to the $\vec{v}_j's$ then $\vec{x}$ is orthogonal to linear combinations of the $\vec{v}_j's$ and hence every vector in W.

(g). True. Suppose that a vector $\vec{w}$ is in W and W^perp. Then $\vec{w}\cdot\vec{w} = 0$ implies $||\vec{w}|| = 0$ which implies $\vec{w} = 0$ (since vectors of zero length are zero).

(h). True by uniqueness of orthogonal projection (a theorem from class).

(i). True. If $(A^T)A = I$, then A is invertible and $A^{-1} = A^T$. Thus $AA^{-1} = AA^T = I$, which means that the columns of A^T are orthonormal and hence the rows of A are orthonormal.

(j). False. $P = \begin{bmatrix} 2 & 0 \\ 0 & 1 \end{bmatrix}$ has orthogonal columns, but P^-1 is not equal to P^T.

(k). True. A theorem from class.

(l) False. Let $\vec{y} = (1,1)$ and $\vec{u} = (0,1)$ Then $\text{proj}_{\text{span}(\vec{u})}(\vec{y}) = \vec{u}$ and $\vec{u}$ is not a multiple of $\vec{y}$.

(m) True by uniqueness of orthogonal decomposition.

3) If $i \neq j$ then $\vec{v}_i \cdot \vec{v}_j = 0$. Since U is orthogonal, the corresponding linear transformation preserves the dot product (angles), thus $(U\vec{v}_i)\cdot(U\vec{v}_j) = \vec{v}_i\cdot\vec{v}_j = 0$. Thus the set $\{U\vec{v}_1, U\vec{v}_2, \dots, U\vec{v}_n\}$ is orthogonal. To see that this set is orthonormal, the same argument as above shows that $||U\vec{v}_i||^2 = (U\vec{v}_i)\cdot(U\vec{v}_i) = \vec{v}_i\cdot\vec{v}_i = ||\vec{v}_i||^2 = 1$ (or just notice that the linear transformation associated with U also preserves lengths).

4) Take $\vec{x} = \vec{e}_i$. Then $U\vec{x}$ is the i^{th} column of U. Take $\vec{y} = e_j$, so that $U\vec{y}$ is the jth column of U. Write $\vec{u}_i$ for the i^{th} column of U for $i = 1, \dots, n$. Then $\vec{u}_i \cdot \vec{u}_j = (Ux)\cdot(U\vec{y}) = \vec{x}\cdot\vec{y} = \vec{e}_i\cdot\vec{e}_j$. Thus $\vec{u}_i\cdot\vec{u}_j$ is zero if $i \neq j$ and is 1 if $i = j$. This means the columns of U are orthonormal, and hence that $(U^T)U = I$ so that $U^{-1} = U^T$.

5) Suppose that $\vec{x}$ and $\vec{y}$ are each orthogonal to the given vector $\vec{v}$, and suppose that c and d are scalars. To show that $c\vec{x} + d\vec{y}$ is in the subspace, one needs to see that $cvecx + d\vec{y}$ is orthogonal to $\vec{v}$. Well, $(c\vec{x}+d\vec{y})\cdot\vec{v} = c(\vec{x}\cdot\vec{v}) + d(\vec{y}\cdot\vec{v}) = c\cdot 0 + d\cdot 0 = 0$ since $\vec{x}\cdot\vec{v} = \vec{y}\cdot\vec{v} = 0$. This shows that $\{\vec{x} : \vec{x}\cdot\vec{v} = 0\}$ is a subspace.

6) a) Let A be the matrix with columns $\vec{v}_1, \vec{v}_2, \vec{v}_3$. One finds that A has two pivots, in the first two columns. So colomn space of A has dimension 2.

b) Note that V is spanned by $\vec{v}_1$ and $\vec{v}_2$. A vector $\vec{x} = (a, b, c)$ is orthogonal to V iff $\vec{x}$ is orthogonal to $\vec{v}_1$ and to $\vec{v}_2$. Thus, $\vec{x}$ is in $V^\perp$ iff $\begin{bmatrix} 1 & 1 & 1 \\ 1 & -2 & -2 \end{bmatrix} \vec{x} = \vec{0}$ So we are reduced to finding the Null space of the above 2x3 matrix. The reduced echelon form of this matrix is $\begin{bmatrix} 1 & 0 & \frac{4}{3} \\ 0 & 1 & \frac{-1}{3} \end{bmatrix}$. So $V^\perp$ is spanned by $(-\frac{4}{3}, \frac{1}{3}, 1)$ or by $(-4, 3, 1)$.

c) $\vec{c}$ is in V iff $\vec{c}$ is equal to its orthogonal projection onto V. Since the orthogonal complement of V is spanned by a single vector, $\vec{c}$ is in V iff $\vec{c}$ is orthogonal to $(-4, 3, 1)$. Thus the condition is that $-4c_1 + c_2 2 + 3c_3 = 0$. If there is a particular solution $\vec{p} = (x, y, z)$, then $\vec{p} + \vec{v}$ is also a solution for any vector $\vec{v}$ in the Nullspace of A. Since the nullspace of A is 1 dimensional there are infinitely many solutions in this case.

(7) (a) The matrix formed by the 3 vectors may be row reduced:

$$\begin{pmatrix} 1 & 1 & 0 \\ 2 & 1 & 2 \\ 1 & 0 & 2 \end{pmatrix} \sim \begin{pmatrix} 1 & 1 & 0 \\ 0 & -1 & 2 \\ 0 & -1 & 2 \end{pmatrix} \sim \begin{pmatrix} 1 & 0 & 2 \\ 0 & -1 & 2 \\ 0 & 0 & 0 \end{pmatrix}$$

hence the dimension of the column space is 2 and is actually spanned by the vetcors $\mathbf{x}_1 = (1, 2, 1)^T$ and $\mathbf{x}_2 = (1, 1, 0)^T$. (b) To find an orthogonal basis $\mathbf{u}_1, \mathbf{u}_2$ with the same span we may take $\mathbf{u}_1 = \mathbf{x}_1$ and

$$\mathbf{u}_2 = \mathbf{x}_2 - \frac{\mathbf{x}_2.\mathbf{x}_1}{\mathbf{x}_1.\mathbf{x}_1}\mathbf{x}_1 = (1, 1, 0)^T - \frac{1}{2}(1, 2, 1)^T = (1/2, 0, -1/2)^T.$$

(8) (a) It is clear that $\mathbf{v}_1.\mathbf{v}_2 = \mathbf{v}_1.\mathbf{v}_3 = \mathbf{v}_2.\mathbf{v}_3 = 0$. (b) an orthogonal set is necessarily linearly independent so the dimension of the span W of these 3 vectors is 3, by the definition of dimension. (c) the projection of a vector $\mathbf{x}$ to the space W is given by

$$\text{proj}_W \mathbf{x} = \frac{\mathbf{x}.\mathbf{v}_1}{\mathbf{v}_1.\mathbf{v}_1}\mathbf{v}_1 + \frac{\mathbf{x}.\mathbf{v}_2}{\mathbf{v}_2.\mathbf{v}_2}\mathbf{v}_2 + \frac{\mathbf{x}.\mathbf{v}_3}{\mathbf{v}_3.\mathbf{v}_3}\mathbf{v}_3 = -\mathbf{v}_1 + 2\mathbf{v}_3 = (-1, -1, 2, -2)^T.$$

(9) (a) The matrix formed by the 3 vectors (as row vectors) can be row reduced to

$$A = \begin{pmatrix} 1 & 1 & 1 \\ 1 & 2 & 2 \\ 1 & 3 & 3 \end{pmatrix} \sim \begin{pmatrix} 1 & 1 & 1 \\ 0 & 1 & 1 \\ 0 & 2 & 2 \end{pmatrix} \sim \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 1 \\ 0 & 0 & 0 \end{pmatrix}.$$

The null space $V^\perp$ is spanned by the vector $(0, 1, -1)^T$. (b) The matrix formed by the 3 vectors (as column vectors) can be row reduced to

$$A = \begin{pmatrix} 1 & 1 & 1 \\ 1 & 2 & 3 \\ 1 & 2 & 3 \end{pmatrix} \sim \begin{pmatrix} 1 & 1 & 1 \\ 0 & 1 & 2 \\ 0 & 0 & 0 \end{pmatrix} \sim \begin{pmatrix} 1 & 0 & -1 \\ 0 & 1 & 1 \\ 0 & 0 & 0 \end{pmatrix}.$$

the equation $A\mathbf{x} = \mathbf{c} = (c_1, c_2, c_3)^T$ is consistent if and only if $\mathbf{c}$ is a linear combination of the vectors $\mathbf{u}_1 = (1, 1, 1), \mathbf{u}_2 = (1, 2, 2)$, i. e.,

$$c_1 = a + b, c_2 = a + 2b, c_3 = a + 2b, a, b \in \mathbf{R}.$$

(c) infinitely many.

(10) The points are $(-1,1),(0,1),(1,0),(2,2)$,

$$A=\begin{pmatrix}1&-1&1\\1&0&0\\1&1&1\\1&2&4\end{pmatrix},\ \mathbf{b}=\begin{pmatrix}1\\1\\0\\2\end{pmatrix},$$

$$A^T=\begin{pmatrix}1&1&1&1\\-1&0&1&2\\1&0&1&4\end{pmatrix},\ A^T\mathbf{b}=\begin{pmatrix}1&1&1&1\\-1&0&1&2\\1&0&1&4\end{pmatrix}\begin{pmatrix}1\\1\\0\\2\end{pmatrix}=\begin{pmatrix}4\\3\\9\end{pmatrix},$$

$$A^TA=\begin{pmatrix}1&1&1&1\\-1&0&1&2\\1&0&1&4\end{pmatrix}\begin{pmatrix}1&-1&1\\1&0&0\\1&1&1\\1&2&4\end{pmatrix}=\begin{pmatrix}4&2&6\\2&6&8\\6&8&18\end{pmatrix}.$$

To find $\hat{\mathbf{x}}$ row reduce the augmented matrix,

$$\begin{pmatrix}4&2&6&4\\2&6&8&3\\6&8&18&9\end{pmatrix}.$$

(11) The vectors are $\mathbf{v}_1=(-1,3,1,1),\mathbf{v}_2=(6,-8,-2,-4),\mathbf{v}_3=(6,3,6,-3)$. First find an orthogonal basis $\mathbf{u}_1,\mathbf{u}_2,\mathbf{u}_3$ with $\mathbf{u}_1=\mathbf{v}_1$,

$$\mathbf{u}_2=(6,-8,-2,-4)+3(-1,3,1,1)=(3,1,1,-1)$$

$$\mathbf{u}_3=(6,3,6,-3)-\frac{1}{2}(-1,3,1,1)-\frac{5}{2}(3,1,1,-1)=(-1,-1,3,-1).$$

Then normalize these vectors:

$$\mathbf{w}_1=\frac{\mathbf{u}_1}{2\sqrt{3}},\mathbf{w}_2=\frac{\mathbf{u}_2}{2\sqrt{3}},\mathbf{w}_3=\frac{\mathbf{u}_3}{2\sqrt{3}}.$$

Now form a matrix Q with $\mathbf{w}_1,\mathbf{w}_2,\mathbf{w}_3$ as column vectors:

$$Q=\frac{1}{2\sqrt{3}}\begin{pmatrix}-1&3&-1\\3&1&-1\\1&1&3\\1&-1&-1\end{pmatrix}$$

Then $R=Q^TA$ where A is the matrix formed by taking the 3 original vectors $\mathbf{v}_1,\mathbf{v}_2,\mathbf{v}_3$ as column vectors:

$$A=\begin{pmatrix}-1&6&6\\3&-8&3\\1&-2&6\\1&-4&-3\end{pmatrix}$$

$$R=\frac{1}{2\sqrt{3}}\begin{pmatrix}-1&3&1&1\\3&1&1&-1\\-1&-1&3&-1\end{pmatrix}\begin{pmatrix}-1&6&6\\3&-8&3\\1&-2&6\\1&-4&-3\end{pmatrix}=\frac{1}{2\sqrt{3}}\begin{pmatrix}12&-36&6\\0&12&30\\0&0&12\end{pmatrix}.$$

(12) First compute

$$A^T A = \begin{pmatrix} 1 & 2 & -1 \\ 2 & 4 & -2 \end{pmatrix} \begin{pmatrix} 1 & 2 \\ 2 & 4 \\ -1 & -2 \end{pmatrix} = \begin{pmatrix} 6 & 12 \\ 12 & 24 \end{pmatrix}$$

$$A^T \mathbf{b} = \begin{pmatrix} 1 & 2 & -1 \\ 2 & 4 & -2 \end{pmatrix} \begin{pmatrix} 3 \\ 2 \\ 1 \end{pmatrix} = \begin{pmatrix} 6 \\ 12 \end{pmatrix}$$

then solve the equation $A^T A\mathbf{x} = A^T\mathbf{b}$

$$\begin{pmatrix} 6 & 12 \\ 12 & 24 \end{pmatrix} \begin{pmatrix} x_1 \\ x_2 \end{pmatrix} = \begin{pmatrix} 6 \\ 12 \end{pmatrix}$$

hence $x_1 = 1, x_2 = 0$.