Math 221
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Name: ______________________________
Wednesday, December 13

**1**. (25 pts – 5 each.) Define the following notions:

**(a).** Linear independence.

**(b).** Orthogonal vectors.

**(c).** Orthogonal projection.

**(d).** Linear transformation.

**(e).** Least-square solution to $A\vec{x} = \vec{b}$.

**2**. (20pts – 2 each.) Mark each of the following True or False. There is no need to justify your answers.

(a) If 3 vectors span $\mathbb{R}^3$ then the vectors are linearly independent.

(b) If an $n \times n$-matrix $A$ is invertible then every $\vec{v} \in \mathbb{R}^n$ is in the column space of $A$.

(c) If $A$ is an $3 \times 4$-matrix then the columns of $A$ are linearly independent.

(d) The map $T : \mathbf{R}^3 \to \mathbf{R}^2$ defined by $T(x, y, z) = (x - y + z, x(y + z))$ is a linear transformation.

(e) For all matrices $A, B, C$, if $AC = BC$ then $A = B$.

(f) rank(A) = dim Nul(A) + dim column space of $A$.

(g) Let $A$ be an $n \times n$ matrix then $A$ is diagonalizable if it has $n$ eigenvalues.

(h) For any $m \times n$-matrix $A$ and any $\vec{b} \in \mathbb{R}^n$ the system $A^T A\vec{x} = A^T\vec{b}$ always has a solution.

(i) The orthogonal projection of $\vec{x}$ onto $V$ is orthogonal to $V$.

(j) If $\hat{x}$ is a least square solution to $A\vec{x} = \vec{b}$ then $\vec{b} - A\hat{x}$ is orthogonal to rowspace of $A$.

**3**. (20pts.) Find a basis for the column space, a basis for the row space, a basis for the null-space, the rank, and the dimension of the null-space of

$$\begin{bmatrix} 1 & -2 & 2 & -1 \\ -3 & 6 & 1 & 10 \\ 1 & -2 & -4 & -7 \\ 1 & -2 & 2 & 0 \end{bmatrix}$$

**4**. (10pts.) Let $A = \begin{bmatrix} 0 & 1 & 0 & 3 \\ 0 & 0 & 1 & -2 \end{bmatrix}$ and $\vec{b} = \begin{bmatrix} -4 \\ 7 \end{bmatrix}$.

**(a).** Find one solution to the linear equation $A\vec{x} = \vec{b}$.

**(b).** Find a basis for the nullspace of $A$.

**(c).** Using the above two parts find *all* the solutions to $A\vec{x} = \vec{b}$.

**5**. (15pts.) Let $\mathbf{b}_1 = \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix}, \mathbf{b}_2 = \begin{bmatrix} -1 \\ 1 \\ 0 \end{bmatrix}, \mathbf{b}_3 = \begin{bmatrix} -3 \\ 1 \\ 1 \end{bmatrix}$.

**(a).** Show that $\mathcal{B} = \{\mathbf{b}_1, \mathbf{b}_2, \mathbf{b}_3\}$ is a basis.

**(b).** Let $x = \begin{bmatrix} 1 \\ -2 \\ 3 \end{bmatrix}$. Find the coordinates of $\mathbf{x}$ with respect to the basis $\mathcal{B}$.

**(c).** Let $V$ be the subspace spanned by $\{\mathbf{b}_2, \mathbf{b}_3\}$. Find an orthonormal basis for $V$.

**6**. (15pts.) Find the least square solution to the system

$$\begin{bmatrix} 1 & 0 \\ 0 & 1 \\ -1 & 1 \\ 0 & 1 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} 0 \\ 1 \\ 0 \\ 1 \end{bmatrix}.$$

**7**. (10pts.) Let $\mathbf{v}_1 = (\frac{2}{3}, -\frac{1}{3}, -\frac{2}{3}), \mathbf{v}_2 = (\frac{1}{\sqrt{2}}, 0, \frac{1}{\sqrt{2}})$.

**(a).** Show that $\mathbf{v}_1, \mathbf{v}_2$ is an orthonormal basis of the span $W = \text{span}\ \{\mathbf{v}_1, \mathbf{v}_2\}$.

**(b).** Let $\mathbf{x} = (2, 1, 3)$ find the distance from $\mathbf{x}$ to $W = \text{span}\ \{\mathbf{v}_1, \mathbf{v}_2\}$.

**8**. (15 pts.) Let $T : \mathbf{R}^4 \to \mathbf{R}^2$ be the linear tranformation $T(x_1, x_2, x_3, x_4) = (x_1 + x_2, x_3 - x_4)$.

**(a).** Find the matrix representing $T$ relative to the standard basis on $\mathbf{R}^4$ and the standard basis on $\mathbf{R}^2$.

**(b).** $\mathcal{B} = \{(1, 1), (1, -1)\}$ is a basis of $\mathbf{R}^2$. Find the matrix representing $T$ relative to the standard basis on $\mathbf{R}^4$ and the basis $\mathcal{B} = \{(1, 1), (1, -1)\}$ on $\mathbf{R}^2$.

**9**. (10pts.) Let $T$ be a linear transformation from $\mathbb{R}^k$ to $\mathbb{R}^k$. Show that the set of $\vec{v} \in \mathbb{R}^k$ such that $T(\vec{v}) = 4\vec{v}$ is a subspace of $\mathbb{R}^k$.

**10**. (10pts.) Let $A$ be an $m \times n$ matrix with *row* vectors $\mathbf{v}_1, \mathbf{v}_2, ..., \mathbf{v}_m$ (so $A = \begin{bmatrix} \mathbf{v}_1 \\ \mathbf{v}_2 \\ \vdots \\ \mathbf{v}_m \end{bmatrix}$) and let $\mathbf{v} = a_1\mathbf{v}_1 + a_2\mathbf{v}_2, ... + a_n\mathbf{v}_m$ be a linear combination of the row vectors. Show that if $\mathbf{v} \neq \mathbf{0}$ then $A\mathbf{v} \neq \mathbf{0}$.