Is $A = 123$
134
$0{-}2-2$ invertible? If so find an inverse. Find all the solutions to $A\vec{x} = \vec{e}_2$.

The matrix 123100
134010
$0{-}2-2001$ row reduces to $10101\frac{3}{2}$
$01100\frac{1}{2}$
$0001-1-\frac{1}{2}$ so A is not invertible. From this we can also see that 1230
1341
$0{-}2-20$ row reduces to 1011
0110
$000-1$ which has a pivot in the last column and therefore $A\vec{x} = \vec{e}_2$ is not consistent (has no solutions).

$B = 120$
$0\frac{1}{2}0$
103 is invertible. Write B and B^{-1} as the product of elementary matrices.

Since we know B is invertible we will reduce B to I keeping track of the elementary row operations. We will write this out as multiplication by elementary matrices.

$$100010-1011200\frac{1}{2}0103 = 1200\frac{1}{2}00-23$$

$$1000200011200\frac{1}{2}00-23 = 1200100-23$$

$$1000100211200100-23 = 120010003$$

$$10001000\frac{1}{3}120010003 = 120010001$$

$$120010001120010001 = 100010001$$

Hence

$$B^{-1} = 12001000110001000\frac{1}{3}1000100211000200011000010-101$$

(You might have a slightly different order with slightly different matrices.) $(B^{-1})^{-1} = B$, the inverse of an elementary matrix is also an elementary matrix and the inverse of AC is $C^{-1}A^{-1}$ so

$$B = 100010-101^{-1}100020001^{-1}100010021^{-1}10001000\frac{1}{3}^{-1}120010001^{-1}$$

$$B = 1000101011000\frac{1}{2}00011000100-211000100031-20010001$$

Assume that AB exists. Show that if the columns of B are linearly dependent then the columns of AB are also.

$AB = A\vec{b}_1\vec{b}_2\ldots\vec{b}_n = A\vec{b}_1A\vec{b}_2\ldots A\vec{b}_n$. The columns of B are linearly dependent iff there are c_i's not all zero such that $c_1\vec{b}_1 + c_1\vec{b}_2\ldots + c_n\vec{b}_n = 0$. Then $A(c_1\vec{b}_1 + c_1\vec{b}_2\ldots + c_n\vec{b}_n) = 0$ and hence $Ac_1\vec{b}_1 + Ac_1\vec{b}_2\ldots + Ac_n\vec{b}_n = c_1A\vec{b}_1 + c_1A\vec{b}_2\ldots + c_nA\vec{b}_n = 0$. Therefore the columns of AB are linearly dependent.

Assume AB exists. If A is invertible and $AB = 0$ what is B? Give an example where $A \neq 0$ and $B \neq 0$ but $AB = 0$.

If A is invertible then we can multiple on the left by A^{-1} to get

$$A^{-1}AB = A0$$

$$IB = 0$$

$$B = 0.$$

In our example neither A nor B can be invertible. Let $A = B = 00$
10. $AB = 00$
1000
$10 = 0$.

Find the determinant of $A = 210-10$
$2-11-30$
$120-23$
02000
12230.

Start by taking the cofactor expansion across the fourth row. Only $a_{4,2}$ is nonzero, so $\det(A) = (2)(-1)^{4+2}\det(A_{4,2}) = (2)\det(A_{4,2})$. Let $B = A_{4,2} =$
$20-10$
$21-30$
$10-23$
1230. To take the determinant of B lets use the cofactor expansion down the last column. Only $b_{3,4}$ is nonzero, so $\det(B) = (3)(-1)^{3+4}\det(B_{3,4}) = -3\det(B_{3,4})$.
$B_{3,4} = 20-1$
$21-3$
123. $\det(B_{3,4}) = 2\det(1-3$
$23) + (-1)\det(21$
$12) = (2)(9) + (-1)(3) = 15$. So $\det(A) = (2)(-3)(15) = -90$.