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# MATH 221 Quiz IV Solutions

**1.** Let $V_n = \{a_0 + a_1 t + a_2 t^2 + ... + a_n t^n \mid a_i \in \mathbf{R}, i = 0, ..., n\}$ be the vector space of polynomials of degree at most $n$. The set $\{1, t, t^2, ..., t^n\}$ is a basis for $V_n$ and is referred to as the standard basis. Consider the case $n = 3$ and show that:

(**a**) the set $\mathcal{B} = \{1, (1-t), (1-t)^2, (1-t)^3\}$ is also a basis of $V_3$;

Solution: *Suppose that* $c_0 + c_1(1-t) + c_2(1-t)^2 + c_3(1-t)^3 = 0$ *then setting* $t = 1$ *yields* $c_0 = 0$ *thus* $c_1(1-t) + c_2(1-t)^2 + c_3(1-t)^3 = 0$. *Factoring out the common factor* $(1-t)$ *yields* $c_1 + c_2(1-t) + c_3(1-t)^2 = 0$. *Setting* $t = 1$ *again yields* $c_1 = 0$ *and hence* $c_2(1-t) + c_3(1-t)^2 = 0$. *Factoring out* $(1-t)$ *gives* $c_2 + c_3(1-t) = 0$ *and setting* $t = 1$ *once more shows that* $c_2 = 0$. *We arrive at* $c_3(1-t)^2 = 0$ *which implies that* $c_3 = 0$.

(**b**) what are the coordinates of the polynomial $1 + t + t^2 + t^3$ relative to the standard basis?

Solution: $[1, 1, 1, 1]$.

(**c**) a polynomial $P$ has coordinates $[1, 1, 1, 1]_{\mathcal{B}}$ relative to $\mathcal{B}$ what are the coordinates of $P$ relative to the standard basis?

Solution: $1 = [1, 0, 0, 0], 1-t = [1, -1, 0, 0], (1-t)^2 = 1-2t+t^2 = [1, -2, 1, 0], (1-t)^3 = 1 - 3t + 3t^2 - t^3 = [1, -3, 3, -1]$. *Writing these as column vectors results in a matrix*

$$P_{\mathcal{B}} = \begin{pmatrix} 1 & 1 & 1 & 1 \\ 0 & -1 & -2 & -3 \\ 0 & 0 & 1 & 3 \\ 0 & 0 & 0 & -1 \end{pmatrix}$$

*and*

$$P_{\mathcal{B}} \begin{pmatrix} 1 \\ 1 \\ 1 \\ 1 \end{pmatrix}_{\mathcal{B}} = \begin{pmatrix} 1 & 1 & 1 & 1 \\ 0 & -1 & -2 & -3 \\ 0 & 0 & 1 & 3 \\ 0 & 0 & 0 & -1 \end{pmatrix} \begin{pmatrix} 1 \\ 1 \\ 1 \\ 1 \end{pmatrix} = \begin{pmatrix} 4 \\ -6 \\ 4 \\ -1 \end{pmatrix}.$$

*Thus the coordinates relative to the standard basis is* $[4, -6, 4, -1]$, *i.e., it is the polynomial* $4 - 6t + 4t^2 - t^3$.

(**d**) what are the coordinates of the polynomial $1 + t + t^2 + t^3$ relative to the basis $\mathcal{B}$ given in (**a**)?

Solution: *We use the formula* $[\mathbf{x}]_{\mathcal{B}} = P_{\mathcal{B}}^{-1}[\mathbf{x}]$. *It is clear that* $P_{\mathcal{B}}^{-1} = P_{\mathcal{B}}$ *hence*

$$[\mathbf{x}]_{\mathcal{B}} = P_{\mathcal{B}}^{-1}[\mathbf{x}] = \begin{pmatrix} 1 & 1 & 1 & 1 \\ 0 & -1 & -2 & -3 \\ 0 & 0 & 1 & 3 \\ 0 & 0 & 0 & -1 \end{pmatrix} \begin{pmatrix} 1 \\ 1 \\ 1 \\ 1 \end{pmatrix} = \begin{pmatrix} 4 \\ -6 \\ 4 \\ -1 \end{pmatrix}_{\mathcal{B}},$$

*i.e.,* $1 + t + t^2 + t^3 = 4 - 6(1-t) + 4(1-t)^2 - (1-t)^3$.

(**e**) the set $\mathcal{C} = \{1, (1+t), (1+t)^2, (1+t)^3\}$ is also a basis for $V_3$ what is the coordinates relative to $\mathcal{C}$ of the polynomial $P = [1, 1, 1, 1]_{\mathcal{B}}$ of part (c)?

Solution: $1 = [1,0,0,0], 1+t = [1,1,0,0], (1+t)^2 = 1+2t+t^2 = [1,2,1,0], (1+t)^3 = 1+3t+3t^2+t^3 = [1,3,3,1]$. *Writing these as column vectors results in a matrix*

$$P_{\mathcal{C}} = \begin{pmatrix} 1 & 1 & 1 & 1 \\ 0 & 1 & 2 & 3 \\ 0 & 0 & 1 & 3 \\ 0 & 0 & 0 & 1 \end{pmatrix}, \; P_{\mathcal{C}}^{-1} = \begin{pmatrix} 1 & -1 & 1 & -1 \\ 0 & 1 & -2 & 3 \\ 0 & 0 & 1 & -3 \\ 0 & 0 & 0 & 1 \end{pmatrix}.$$

*We know that*

$$[\mathbf{x}]_{\mathcal{C}} = P_{\mathcal{C}\leftarrow\mathcal{B}} \begin{pmatrix} 1 \\ 1 \\ 1 \\ 1 \end{pmatrix}_{\mathcal{B}} = P_{\mathcal{C}}^{-1} P_{\mathcal{B}} \begin{pmatrix} 1 \\ 1 \\ 1 \\ 1 \end{pmatrix}_{\mathcal{B}}$$

*because* $P_{\mathcal{C}\leftarrow\mathcal{B}} = P_{\mathcal{C}}^{-1} P_{\mathcal{B}}$ *and*

$$P_{\mathcal{C}}^{-1} P_{\mathcal{B}} \begin{pmatrix} 1 \\ 1 \\ 1 \\ 1 \end{pmatrix}_{\mathcal{B}} = \begin{pmatrix} 1 & -1 & 1 & -1 \\ 0 & 1 & -2 & 3 \\ 0 & 0 & 1 & -3 \\ 0 & 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} 4 \\ -6 \\ 4 \\ -1 \end{pmatrix} = \begin{pmatrix} 15 \\ -17 \\ 7 \\ -1 \end{pmatrix}_{\mathcal{C}}$$

**2.** Given a polynomial $P(t) = a_0 + a_1 t + a_2 t^2 + \ldots + a_n t^n$ of degree $n$ then its derivative $P^{'}(t) = a_1 + 2a_2 t + \ldots + n a_n t^{n-1}$ is a polynomial of degree at most $n-1$. Consider the case $n = 3$ and show that:

(**a**) the map $T : V_3 \to V_2$ defined by $T(P(t)) = P^{'}(t)$ is a linear transformation;

Solution: *If* $P(t)$ *and* $Q(t)$ *are two polynomials of degree at most* 3 *then* $P(t) + Q(t)$ *is also a polynomial of degree* 3 *then* $T(P(t)+Q(t)) = (P(t)+Q(t))^{'} = P^{'}(t) + Q^{'}(t) = T(P(t)) + T(Q(t))$. *If* $c$ *is a scalar then* $T(cP(t)) = (cP(t))^{'} = cP^{'}(t) = cT(P(t))$.

(**b**) write down the matrix $A$ representating the transformation $T$ relative to the standard bases for $V_3$ and $V_2$?

Solution: *The standard basis for* $V_3 = \{e_1 = 1, e_2 = t, e_3 = t^2, e_4 = t^3\}$ *and thot of* $V_2 = \{e_1 = 1, e_2 = t, e_3 = t^2\}$. *We have*

$$\begin{aligned} Te_1 &= (1)^{'} = 0 = 0(1) + 0(t) + 0(t^2) = [0,0,0] \\ Te_2 &= (t)^{'} = 1 = 1(1) + 0(t) + 0(t^2) = [1,0,0] \\ Te_3 &= (t^2)^{'} = 2t = 0(1) + 2(t) + 0(t^2) = [0,2,0] \\ Te_4 &= (t^3)^{'} = 3t^2 = 0(1) + 0(t) + 3(t^2) = [0,0,3] \end{aligned}$$

*thus the matrix* $A$ *is given by*

$$A = \begin{pmatrix} 0 & 1 & 0 & 0 \\ 0 & 0 & 2 & 0 \\ 0 & 0 & 0 & 3 \end{pmatrix}$$

(**c**) what is the null space of the matrix $A$ in (**b**)?

Solution: *Let* $\mathbf{x} = [x_1, x_2, x-3, x_4]$ *be a vector in the null space of* $A$ *then* $A\mathbf{x} = 0$ *which is equivalent to the condition that* $x_2 = x_3 = x_4 = 0$ *with* $x_1$ *being a free variable so:*

$$\operatorname{nul} A = \{x_1 \begin{pmatrix} 1 \\ 0 \\ 0 \\ 0 \end{pmatrix}\}$$

(**d**) what is the column space of the matrix $A$ in (**b**)?

Solution: *The column space is the linear span of the vectors*

$$\{\begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix}, \begin{pmatrix} 0 \\ 2 \\ 0 \end{pmatrix}, \begin{pmatrix} 0 \\ 0 \\ 3 \end{pmatrix}\}$$

*which is all of* $V_2$.

(**e**) verify the Rank Theorem by using your answers for (**c**) and (**d**).

Solution: *By (c) the nullity of* $A = \dim \operatorname{nul} A = 1$ *and by (d) the rank of* $A = \dim \text{column space} = 3$ *thus nullity of* $A$ *+ rank of* $A = 4 =$ *number of columns of* $A$ *verifying the Rank Theorem.*