Let V be a subspace of $\mathbf{R}^4$ whose basis is $\vec{x_1} = (1, 0, 1, 0, 0)$, $\vec{x_2} = (0, 1, -1, -1, 0)$ and $\vec{x_3} = (0, 0, 10, 0, 10)$. Find an orthogonal basis B for V. (Here it might be useful to multiply one of the vectors you get to avoid fractional entries.) Find an orthonormal basis for V. Find a basis for $V^\perp$. (Hint: The orthogonal complement of the rowspace of a matrix A is what?) Find an orthogonal basis for $V^\perp$. Expand the basis B found in the first part to an orthogonal basis of $\mathbb{R}^5$. (This does not involve any computations.) Let $\vec{x} = (1, 2, 1, 0, 1)$. Find $proj_V(\vec{x})$, $\vec{z} = \vec{x} - proj_V(\vec{x})$ and $[proj_V(\vec{x})]_B$. (Expect for part (b) all the numbers work out nicely.)

(a) Let $\vec{v}_1 = \vec{x}_1$. Then $\vec{v}_2 = \vec{x}_2 - proj_{W_1}(\vec{x_2}) = \vec{x}_2 - \frac{\vec{v}_1 \cdot \vec{x}_2}{\vec{v}_1 \cdot \vec{v}_1}\vec{v}_1$, where $W_1 = Span\{\vec{v}_1\}$. So $\vec{v}_2 = (0, 1, -1, -1, 0) - \frac{-1}{2}(1, 0, 1, 0, 0) = (\frac{1}{2}, 1, -\frac{1}{2}, -1, 0)$. To make life easier lets multiply $\vec{v}_2$ by 2 to get $\vec{v}_2 = (1, 2, -1, -2, 0)$. Then $\vec{v}_3 = \vec{x}_3 - proj_{W_2}(\vec{x_3}) = \vec{x}_3 - \frac{\vec{v}_1 \cdot \vec{x}_3}{\vec{v}_1 \cdot \vec{v}_1}\vec{v}_1 - \frac{\vec{v}_2 \cdot \vec{x}_3}{\vec{v}_2 \cdot \vec{v}_2}\vec{v}_2$, where $W_2 = Span\{\vec{v}_1, \vec{v}_1\}$. Hence $\vec{v}_3 = (0, 0, 10, 0, 10) - \frac{10}{2}(1, 0, 1, 0, 0) - \frac{-10}{10}(1, 2, -1, -2, 0) = (0, 0, 10, 0, 10) - (5, 0, 5, 0, 0) + (1, 2, -1, -2, 0) = (-4, 2, 4, -2, 10)$. Hence an orthogonal basis is for V is $(1, 0, 1, 0, 0)$, $(1, 2, -1, -2, 0)$, $(-4, 2, 4, -2, 10)$.

(b) Just normalize the above basis to get $(\frac{1}{\sqrt{2}}, 0, \frac{1}{\sqrt{2}}, 0, 0)$, $\frac{1}{\sqrt{10}}(1, 2, -1, -2, 0)$, $\frac{1}{\sqrt{140}}(-4, 2, 4, -2, 10)$.

(c) Let A be the matrix whose rows are the above vectors, $A = 10100$
$01 - 1 - 10$
0010010
. So V is the row space of A and $V^\perp$ is the nullspace of A. The reduced row echelon form of A is 10001
$010 - 11$
$0010 - 1$
. So a basis for $V^\perp$ are the two vectors $\vec{x}_4 = (0, 1, 0, 1, 0)$ and $\vec{x}_5 = (1, -1, -1, 0, 1)$.

(d). We will use $\vec{v}_4$ and $\vec{v}_5$ rather than $\vec{v}_1$ and $\vec{v}_2$. Let $\vec{v}_4 = \vec{x}_4 = (0, 1, 0, 1, 0)$. Then $\vec{v}_5 = \vec{x}_5 - proj_{W_4}(\vec{x_5}) = \vec{x}_5 - \frac{\vec{v}_4 \cdot \vec{x}_5}{\vec{v}_4 \cdot \vec{v}_4}\vec{v}_4$, where $W_4 = Span\{\vec{v}_4\}$. So $\vec{v}_2 = (1, -1, -1, 0, 1) - \frac{-1}{2}(0, 1, 0, 1, 0) = (1, -\frac{1}{2}, -1, \frac{1}{2}, 1)$. Again to make life easier lets multiply $\vec{v}_2$ by 2 to get $\vec{v}_2 = (2, -1, -2, 1, 2)$.

(e) Since the element of the basis in part (d) is orthogonal to the elements in the basis in part (a). The union of the two basis is the desired basis. $(1, 0, 1, 0, 0)$, $(1, 2, -1, -2, 0)$, $(-4, 2, 4, -2, 10)$, $(0, 1, 0, 1, 0)$, $(2, -1, -2, 1, 2)$.

(f) $proj_V(\vec{x}) = \frac{\vec{v}_1 \cdot \vec{x}}{\vec{v}_1 \cdot \vec{v}_1}\vec{v}_1 + \frac{\vec{v}_2 1 \cdot \vec{x}}{\vec{v}_2 \cdot \vec{v}_2}\vec{v}_2 + \frac{\vec{v}_3 1 \cdot \vec{x}}{\vec{v}_3 \cdot \vec{v}_3}\vec{v}_3 = \frac{2}{2}\vec{v}_1 + \frac{4}{10}\vec{v}_2 + \frac{14}{140}\vec{v}_3 = (1, 1, 1, -1, 1)$. Hence $[proj_V(\vec{x})]_B = (1, \frac{1}{5}, \frac{1}{10})$. $\vec{z} = (1, 2, 1, 0, 1) - (1, 1, 1, -1, 1) = (0, 1, 0, 1, 0)$.