Math 221 – Linear Algebra
Exam 1

1. (20 points – 4 each) TRUE/FALSE. Determine whether the followings statements are true or false. (Comment: no reason needed.)

1.1. Let $T : \mathbb{R}^n \to \mathbb{R}^m$ be a linear transformation. Then there is a $n \times m$ matrix A such $T(\vec{x}) = A\vec{x}$.

1.2. Let $T : \mathbb{R}^m \to \mathbb{R}^n$ be a linear transformation. Let $S = \{\vec{v}_1, \ldots, \vec{v}_n\}$ be a set of linearly independent vectors in $\mathbb{R}^m$. Then the set of vectors $\{T(\vec{v}_1), \ldots, T(\vec{v}_n)\}$ is also linearly independent.

1.3. Let A and B be two invertible matrices. Then ABA^{-1} is invertible.

1.4. If $BA = I$ then both A and B are invertible.

1.5. Let $T(\vec{x})$ be a linear transformation. T is one to one if and only if the kernel of T is $\{\vec{0}\}$.

1.6. If A is a 3×4 matrix of rank 2 then the dimension of the kernel of A is 1.

1.7. If A and B are square matrices then $(A+B)^2 = A^2 + 2AB + B^2$

1.8. Let A be an $m \times n$ matrix, if the rank of A is m then the linear transformation $A\vec{x}$ is onto.

1.9. Let $V = Span\{\vec{v_1}, \vec{v_2}, \vec{v_3}, \vec{v_4}\}$ be a subspace of $\mathbb{R}^4$ of dimension 3. Then $\{\vec{v_1}, \vec{v_2}, \vec{v_3}, \vec{v_4}\}$ form a basis of V.

1.10. Let V be a subspace of $\mathbb{R}^n$. A basis of V is a set of linearly independent vectors in V that is as large as possible.

2. (15 points)

2.1. Write $\begin{bmatrix} 1 \\ 4 \end{bmatrix}$ as a linear combination of $\begin{bmatrix} 1 \\ 1 \end{bmatrix}$ and $\begin{bmatrix} -1 \\ 2 \end{bmatrix}$.

2.2. Let $T : \mathbb{R}^2 \to \mathbb{R}^2$ be a linear transformation. If $T(\begin{bmatrix} 1 \\ 1 \end{bmatrix}) = (\begin{bmatrix} 2 \\ 3 \end{bmatrix}), T(\begin{bmatrix} -1 \\ 2 \end{bmatrix}) = \begin{bmatrix} 4 \\ 5 \end{bmatrix}$ then $T(\begin{bmatrix} 1 \\ 4 \end{bmatrix})$ =?.

3. (30 points)

Let $A = \begin{bmatrix} 1 & -1 & 2 & 1 & -1 \\ -2 & 2 & -4 & -1 & 3 \\ 3 & -2 & 6 & 3 & 1 \\ 4 & -4 & 8 & 4 & -4 \end{bmatrix}$, $\vec{b_1} = \begin{bmatrix} -2 \\ 8 \\ 4 \\ -8 \end{bmatrix}$ and $\vec{b_2} = \begin{bmatrix} 0 \\ 2 \\ 4 \\ 0 \end{bmatrix}$

3.1. Row reduce the matrix $[A|\vec{b_1}|\vec{b_2}]$ into reduced row echelon form (this is not that hard).

3.2. Find a basis for $\ker(A)$. What is the dimension of $\ker(A)$?

3.3. Find a basis for the Image(A). What is the dimension of Image(A)?

3.4. The vectors $\vec{b_1}$ and $\vec{b_2}$ are the Image(A). Why?

3.5. Find all solutions of $A\vec{x} = \vec{b_1}$.

3.6. Find all solutions of $A\vec{x} = \vec{b_2}$.

3.7. Use your answers to describe *all* possible matrices C with the property that $AC = \begin{bmatrix} -2 & 0 \\ 8 & 2 \\ 4 & 4 \\ -8 & 0 \end{bmatrix}$.

4. (15 points) **4.1.** Write down the standard matrix A for the linear transformation $T : \mathbb{R}^3 \longrightarrow \mathbb{R}^3$ which is a projection on the $y-z$ plane followed by a counterclockwise rotation of an angle of $\frac{\pi}{3}$ (60 degrees) in the $y-z$ plane (i.e., find A such that $T(\vec{x}) = A\vec{x}$).

4.2. What is the image of T (describe this geometrically)? What is the dimension of the image of T? What is the rank of T? (Hint: you should need to do any computation or row reduction to answer this question or any of the following questions.)

4.3. What is the dimension of the kernel of T? What is the image of T (describe this geometrically)?

4.4. Is T 1-1? Why or why not? Is T onto? Why or why not?

5. (20 points – 4 each) Provide the precise definition of each of the following concept:

5.1. A vector $\vec{x}$ is a *linear combination* of the vectors $\vec{v}_1, \ldots, \vec{v}_n$.

5.2. The *span* of a set of vectors $\{\vec{v}_1, \ldots, \vec{v}_n\}$.

5.3. A set of vectors $\{\vec{v}_1, ..., \vec{v}_n\}$ being *linearly independent*.

5.4. A function $T : X \to Y$ is *onto*.

5.5. A is an *invertible matrix*.