Math 221 – Linear Algebra
Exam 1

1. (20 points – 4 each) TRUE/FALSE. Determine whether the followings statements are true or false. (Comment: no reason needed.)

1.1. Let $T : \mathbb{R}^n \to \mathbb{R}^m$ be a linear transformation. Then there is a $n \times m$ matrix A such $T(\vec{x}) = A\vec{x}$.

FALSE. A must be $m \times n$.

1.2. Let $T : \mathbb{R}^m \to \mathbb{R}^n$ be a linear transformation. Let $S = \{\vec{v}_1, \dots, \vec{v}_n\}$ be a set of linearly independent vectors in $\mathbb{R}^m$. Then the set of vectors $\{T(\vec{v}_1), \dots, T(\vec{v}_n)\}$ is also linearly independent.

FALSE. Consider $T : \mathbb{R}^m \to \mathbb{R}^n$ by $T(\vec{x}) = \vec{0}$.

1.3. Let A and B be two invertible matrices. Then ABA^{-1} is invertible.

TRUE. $(AB^{-1}A^{-1})(ABA^{-1}) = I$.

1.4. If $BA = I$ then both A and B are invertible.

FALSE. $A = \begin{bmatrix} 2 & -1 \\ -1 & 1 \\ 0 & 0 \end{bmatrix}$ and $B = \begin{bmatrix} 1 & 1 & 1 \\ 1 & 2 & 3 \end{bmatrix}$ then $BA = I_2$.

1.5. Let $T(\vec{x})$ be a linear transformation. T is one to one if and only if the kernel of T is $\{\vec{0}\}$.

TRUE. Assume that $T(\vec{x}) = A\vec{x}$. Then kernel of A is $\{\vec{0}\}$ iff there are no free variables iff $A\vec{x} = \vec{b}$ has at most one solution for all $\vec{b}$ iff T is one to one.

1.6. If A is a 3×4 matrix of rank 2 then the dimension of the kernel of A is 1.

FALSE. By the rank theorem the dimension of the kernel is the number of columns (4) minus the rank (2), so 2.

1.7. If A and B are square matrices then $(A+B)^2 = A^2 + 2AB + B^2$

FALSE. You need to have AB=BA for this to be true which is not usually the case. Take $A = \begin{bmatrix} 1 & 1 \\ 1 & 0 \end{bmatrix}$ and $B = \begin{bmatrix} 1 & 1 \\ 0 & 1 \end{bmatrix}$.

1.8. Let A be an $m \times n$ matrix, if the rank of A is m then the linear transformation $A\vec{x}$ is onto.

TRUE. If the rank of A is m then $A\vec{x} = \vec{b}$ is consistent for all $\vec{b}$ which means that $A\vec{x}$ is onto.

1.9. Let $V = Span\{\vec{v_1}, \vec{v_2}, \vec{v_3}, \vec{v_4}\}$ be a subspace of $\mathbb{R}^4$ of dimension 3. Then $\{\vec{v_1}, \vec{v_2}, \vec{v_3}, \vec{v_4}\}$ form a basis of V.

FALSE. A basis of V has only 3 vectors in it.

1.10. Let V be a subspace of $\mathbb{R}^n$. A basis of V is a set of linearly independent vectors in V that is as large as possible.

TRUE. The dimension of V is m which less than or equal to n. There are at most m linearly independent vectors in V and m linearly independent vectors in V forms a basis of V.

2. (15 points)

2.1. Write $\begin{bmatrix} 1 \\ 4 \end{bmatrix}$ as a linear combination of $\begin{bmatrix} 1 \\ 1 \end{bmatrix}$ and $\begin{bmatrix} -1 \\ 2 \end{bmatrix}$.

We must solve the equation $x_1 \begin{bmatrix}1\\1\end{bmatrix} + x_2 \begin{bmatrix}-1\\2\end{bmatrix} = \begin{bmatrix}1\\4\end{bmatrix}$. The augmented matrix of this system is $\begin{bmatrix}1 & -1 & 1\\1 & 2 & 4\end{bmatrix}$ which row-reduces to $\begin{bmatrix}1 & 0 & 2\\0 & 1 & 1\end{bmatrix}$. Hence $2\begin{bmatrix}1\\1\end{bmatrix} + \begin{bmatrix}-1\\2\end{bmatrix} = \begin{bmatrix}1\\4\end{bmatrix}$.

2.2. Let $T : \mathbb{R}^2 \to \mathbb{R}^2$ be a linear transformation. If $T(\begin{bmatrix}1\\1\end{bmatrix}) = (\begin{bmatrix}2\\3\end{bmatrix}), T(\begin{bmatrix}-1\\2\end{bmatrix}) = \begin{bmatrix}4\\5\end{bmatrix}$ then $T(\begin{bmatrix}1\\4\end{bmatrix})$ =?.

$T(\begin{bmatrix}1\\4\end{bmatrix}) = T(2\begin{bmatrix}1\\1\end{bmatrix} + \begin{bmatrix}-1\\2\end{bmatrix}) = 2T(\begin{bmatrix}1\\1\end{bmatrix}) + T(\begin{bmatrix}-1\\2\end{bmatrix}) = 2\begin{bmatrix}2\\3\end{bmatrix} + \begin{bmatrix}4\\5\end{bmatrix} = \begin{bmatrix}4\\6\end{bmatrix} + \begin{bmatrix}4\\5\end{bmatrix} = \begin{bmatrix}8\\11\end{bmatrix}$, since T is a linear transformation.

3. (30 points)

Let $A = \begin{bmatrix}1 & -1 & 2 & 1 & -1\\-2 & 2 & -4 & -1 & 3\\3 & -2 & 6 & 3 & 1\\4 & -4 & 8 & 4 & -4\end{bmatrix}$, $\vec{b_1} = \begin{bmatrix}-2\\8\\4\\-8\end{bmatrix}$ and $\vec{b_2} = \begin{bmatrix}0\\2\\4\\0\end{bmatrix}$

3.1. Row reduce the matrix $[A|\vec{b}_1|\vec{b}_2]$ into reduced row echelon form (this is not that hard).

3.2. Find a basis for ker(A). What is the dimension of ker(A)?

3.3. Find a basis for the Image(A). What is the dimension of Image(A)?

3.4. The vectors $\vec{b}_1$ and $\vec{b}_2$ are the Image(A). Why?

3.5. Find all solutions of $A\vec{x} = \vec{b_1}$.

3.6. Find all solutions of $A\vec{x} = \vec{b_2}$.

3.7. Use your answers to describe *all* possible matrices C with the property that $AC = \begin{bmatrix}-2 & 0\\8 & 2\\4 & 4\\-8 & 0\end{bmatrix}$.

The matrix $[A|\vec{b}_1|\vec{b}_2]$ reduces to $B = \begin{bmatrix}1 & 0 & 2 & 0 & 2 & 4 & 2\\0 & 1 & 0 & 0 & 4 & 10 & 4\\0 & 0 & 0 & 1 & 1 & 4 & 2\\0 & 0 & 0 & 0 & 0 & 0 & 0\end{bmatrix}$.

Hence a basis for the kernel are the two vectors $\vec{x}_1 = \begin{bmatrix}-2\\0\\1\\0\\0\end{bmatrix}$ and $\vec{x}_2 = \begin{bmatrix}-2\\-4\\0\\-1\\1\end{bmatrix}$. ker($A$) has dimension 2.

The pivot columns of rref(A) are the first, second and fourth. So the first, second and fourth columns of A are a basis for the Image(A). Image(A) has dimension 3.

In rref($[A|\vec{b}_1|\vec{b}_2]$) the last two columns are a linear combination of the pivot columns and hence $\vec{b}_1$ and $\vec{b}_2$ are linear combinations of the columns of A.

A solution for $A\vec{x} = \vec{b_1}$ is $\vec{x}_p = \begin{bmatrix}4\\10\\0\\4\\0\end{bmatrix}$ and all the solutions are in the form $\vec{x}_p + \vec{x}$, where $\vec{x}$ is in the kernel of A. Or a solution looks like $\vec{x}_p + c_1\vec{x}_1 + c_2\vec{x}_2$ for c_1 and c_2 scalars. Let $H_1 = \vec{x}_p + c_1\vec{x}_1 + c_2\vec{x}_2$.

A solution for $A\vec{x} = \vec{b_2}$ is $\vec{x}_q = \begin{bmatrix} 2 \\ 4 \\ 0 \\ 2 \\ 0 \end{bmatrix}$ and all the solutions are in the form $\vec{x}_q + \vec{x}$, where $\vec{x}$ is in the kernel of A. Or a solution looks like $\vec{x}_q + c_1\vec{x}_1 + c_2\vec{x}_2$ for c_1 and c_2 scalars. Let $H_2 = \vec{x}_q + c_1\vec{x}_1 + c_2\vec{x}_2$.

Then $C = [\vec{u}_1\ \vec{u}_2]$, where $\vec{u}_i \in H_i$.

4. (15 points) **4.1.** Write down the standard matrix A for the linear transformation $T : \mathbb{R}^3 \longrightarrow \mathbb{R}^3$ which is a projection on the $y - z$ plane followed by a counterclockwise rotation of an angle of $\frac{\pi}{3}$ (60 degrees) in the $y - z$ plane (i.e., find A such that $T(\vec{x}) = A\vec{x}$).

$A = [T(\vec{e}_1)\ T(\vec{e}_2)\ T(\vec{e}_3)]$. $\vec{e}_1$ is projected to $\vec{0}$ which is rotated to $\vec{0} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$. $\vec{e}_2$ is projected to itself $\vec{e}_2$ which is then rotated to $\begin{bmatrix} 0 \\ 1/2 \\ \sqrt{3}/2 \end{bmatrix}$. $\vec{e}_3$ is projected to itself $\vec{e}_3$ which is then rotated to $\begin{bmatrix} 0 \\ -\sqrt{3}/2 \\ 1/2 \end{bmatrix}$. So

$$A = \begin{bmatrix} 0 & 0 & 0 \\ 0 & 1/2 & -\sqrt{3}/2 \\ 0 & \sqrt{3}/2 & 1/2 \end{bmatrix}$$

4.2. What is the image of T (describe this geometrically)? What is the dimension of the image of T? What is the rank of T? (Hint: you should need to do any computation or row reduction to answer this question or any of the following questions.)

Clearly the image of T is the $y - z$ plane which has dimension 2. So A must have rank 2.

4.3. What is the dimension of the kernel of T? What is the image of T (describe this geometrically)?

By the rank-nullity theorem the dimension of the kernel of T must be 1. The kernel is the x-axis which has dimension 1.

4.4. Is T 1-1? Why or why not? Is T onto? Why or why not?

T is not 1-1 because the kernel of T is not the zero vector. T is not onto because the image of T is not all $\mathbb{R}^3$.

5. (20 points – 4 each) Provide the precise definition of each of the following concept:

5.1. A vector $\vec{x}$ is a *linear combination* of the vectors $\vec{v}_1, \ldots, \vec{v}_n$.

If there are scalars c_i's such that $\vec{x} = c_1\vec{v}_1 + \ldots + c_n\vec{v}_n$.

5.2. The *span* of a set of vectors $\{\vec{v}_1, \ldots, \vec{v}_n\}$.

All linear combinations of the vectors $\{\vec{v}_1, \ldots, \vec{v}_n\}$.

5.3. A set of vectors $\{\vec{v}_1, ..., \vec{v}_n\}$ being *linearly independent*.

No vector $\vec{v}_j$ is a linear combination of the others.

5.4. A function $T : X \to Y$ is *onto*.

If for all $y \in Y$, there is a $x \in X$ such that $T(x) = y$.

5.5. A is an *invertible matrix*.

There is a matrix A^{-1} such that $AA^{-1} = I = A^{-1}A$. This implies that A must be square.