Math 221 – Linear Algebra
Quiz 1

1. (10 points) Consider the following system of linear equations:

$$\begin{array}{ccccccc} 2x_1 & + & 4x_2 & + & 6x_3 & = & 6 \\ x_1 & + & 3x_2 & + & 4x_3 & = & 4 \\ & & x_2 & + & ax_3 & = & 1 \end{array}$$

1.1. (1 point) Find A and $\vec{b}$ such that the above system is in the form $A\vec{x} = \vec{b}$ (i.e. an matrix equation).

$$A = \begin{bmatrix} 2 & 4 & 6 \\ 1 & 3 & 4 \\ 0 & 1 & a \end{bmatrix} \text{ and } \vec{b} = \begin{bmatrix} 6 \\ 4 \\ 1 \end{bmatrix}.$$

1.2. (5 points) Explicitly showing *all* the steps reduce the *augmented matrix* into reduce echelon form. (Hint: there are 2 cases for the rest of the problem.)

The augmented matrix is $M = \begin{bmatrix} 2 & 4 & 6 & 6 \\ 1 & 3 & 4 & 4 \\ 0 & 1 & a & 1 \end{bmatrix}$. First lets divide the top row by 2 to get: $\begin{bmatrix} 1 & 2 & 3 & 3 \\ 1 & 3 & 4 & 4 \\ 0 & 1 & a & 1 \end{bmatrix}$. Now multiple the top row by -1 and add to the second row to get $\begin{bmatrix} 1 & 2 & 3 & 3 \\ 0 & 1 & 1 & 1 \\ 0 & 1 & a & 1 \end{bmatrix}$.

Subtract the second row from the third row to get $\begin{bmatrix} 1 & 2 & 3 & 3 \\ 0 & 1 & 1 & 1 \\ 0 & 0 & a-1 & 0 \end{bmatrix}$.

Now there are two cases; if $a \neq 1$ or $a = 1$.

First lets assume $a \neq 1$. Then we can divide the last row by $a - 1 \neq 0$ to get: $\begin{bmatrix} 1 & 2 & 3 & 3 \\ 0 & 1 & 1 & 1 \\ 0 & 0 & 1 & 0 \end{bmatrix}$.

Then subtract 3 times the last row from the first row and the last row from the second row to get: $\begin{bmatrix} 1 & 2 & 0 & 3 \\ 0 & 1 & 0 & 1 \\ 0 & 0 & 1 & 0 \end{bmatrix}$. Now subtract 2 times the second from the first row to get: $\begin{bmatrix} 1 & 0 & 0 & 1 \\ 0 & 1 & 0 & 1 \\ 0 & 0 & 1 & 0 \end{bmatrix}$ which is in reduce echelon form.

Now assume $a = 1$. Hence $N = \begin{bmatrix} 1 & 2 & 3 & 3 \\ 0 & 1 & 1 & 1 \\ 0 & 0 & 0 & 0 \end{bmatrix}$. Subtract two times the second from the first row to get $\begin{bmatrix} 1 & 0 & 1 & 1 \\ 0 & 1 & 1 & 1 \\ 0 & 0 & 0 & 0 \end{bmatrix}$ which is in reduce echelon form.

1.3. (1 point) What is the rank of A?

If $a = 1$ then the rank of A is 2; otherwise A rank is 3.

1.4. (1 point) Is there an a such that the system does not have a solution? Why or why not?

No! It is never the case that the last column contains a leading 1 and hence it is always consistent.

1.5. (2 points) Find the general solution (in vector form) for the above system.

There are two cases: either the reduce echelon from is $\begin{bmatrix}1&0&0&1\\0&1&0&1\\0&0&1&0\end{bmatrix}$ or $\begin{bmatrix}1&0&1&1\\0&1&1&1\\0&0&0&0\end{bmatrix}$. In the first case, the general solution is $x_1 = 0, x_2 = 1$ and $x_2 = 1$ or $\vec{x} = \begin{bmatrix}1\\1\\0\end{bmatrix}$. In the second case, the general solution is $x_1 = 1 - x_3, x_2 = 1 - x_3$ and x_3 is free. Another way to write this is: $\vec{x} = \begin{bmatrix}1-x_3\\1-x_3\\x_3\end{bmatrix} = \begin{bmatrix}1\\1\\0\end{bmatrix} + \begin{bmatrix}-x_3\\-x_3\\x_3\end{bmatrix} = \begin{bmatrix}1\\1\\0\end{bmatrix} + x_3\begin{bmatrix}-1\\-1\\1\end{bmatrix} = \begin{bmatrix}1\\1\\0\end{bmatrix} + t\begin{bmatrix}-1\\-1\\1\end{bmatrix}$.

2. (6 points) Let $\vec{x}_1 = \begin{bmatrix}1\\-2\\4\end{bmatrix}$, $\vec{x_2} = \begin{bmatrix}3\\2\\1\end{bmatrix}$ and $\vec{y} = \begin{bmatrix}4\\0\\b\end{bmatrix}$.

2.1. Find a value of b such that $\vec{y}$ *is* a linear combination of $\vec{x}_1$ and $\vec{x_2}$.

$\vec{y}$ is a linear combination of $\vec{x}_1$ and $\vec{x_2}$ if and only if the linear system associated to the augmented matrix $\begin{bmatrix}1&3&4\\-2&2&0\\4&1&b\end{bmatrix}$ is consistent.

Let us row reduce the matrix, to find a b for which the linear system is consistent. Multiply by 2 the first row and add it to the second to get: $\begin{bmatrix}1&3&4\\0&8&8\\4&1&b\end{bmatrix}$. Now multiple the top row by -4 and add it to the third to get $\begin{bmatrix}1&3&4\\0&8&8\\0&-11&-16+b\end{bmatrix}$. Divide by 8 the second row to get $\begin{bmatrix}1&3&4\\0&1&1\\0&-11&-16+b\end{bmatrix}$. Multiply the second row by -3 and add it to the first row to get $\begin{bmatrix}1&0&1\\0&1&1\\0&-11&-16+b\end{bmatrix}$. Multiply the second row by 11 and add it to the third row to get $\begin{bmatrix}1&0&4\\0&1&1\\0&0&-5+b\end{bmatrix}$. Now, the system is consistent if and only if $-5 + b = 0$. Hence if $b = 5$.

2.2. Find a value of b such that $\vec{y}$ *is not* a linear combination of $\vec{x}_1$ and $\vec{x_2}$.

Take any $b \neq 5$.

3. (4 points) Both of the following statements are false. Provide an example.

3.1. (2 point) If A is a 5×4 matrix and has rank 4 then for all $\vec{b} \in \mathbb{R}^4$, $A\vec{x} = \vec{b}$ is consistent.

Take $A = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 \end{bmatrix}$ and $\vec{b} = \begin{bmatrix} 0 \\ 0 \\ 0 \\ 0 \\ 1 \end{bmatrix}$.

3.2. (2 points) If A is an upper triangular matrix then the rank of A is the number of non-zero entries on A's diagonal.

$$A = \begin{bmatrix} 1 & 1 & -1 \\ 0 & 0 & 1 \\ 0 & 0 & 0 \end{bmatrix}$$