Math 221 – Linear Algebra
Quiz 1

1. (This is based on problem 76 on page 90.) The color of light can be represented in a vector, $\begin{bmatrix} R \\ G \\ B \end{bmatrix}$, where R is the amount of red light, G is the amount of green light and B is the amount of blue light. The human eye and the brain transforms the incoming signal into the signal $\begin{bmatrix} I \\ L \\ S \end{bmatrix}$, where the intensity $I = \frac{R+G+B}{3}$, the long-wave signal $L = R - G$ and the short wave signal $S = B - \frac{R+G}{2}$.

1.1. Find the matrix $\mathcal{A}$ of the transformation taking $\begin{bmatrix} R \\ G \\ B \end{bmatrix}$ to $\begin{bmatrix} I \\ L \\ S \end{bmatrix}$.

$\mathcal{A} = \begin{bmatrix} \frac{1}{3} & \frac{1}{3} & \frac{1}{3} \\ 1 & -1 & 0 \\ -\frac{1}{2} & -\frac{1}{2} & 1 \end{bmatrix}$.

1.2. Find $\mathcal{A}^{-1}$.

$\mathcal{A}^{-1} = \begin{bmatrix} 1 & \frac{1}{2} & -\frac{1}{3} \\ 1 & -\frac{1}{2} & -\frac{1}{3} \\ 1 & 0 & \frac{2}{3} \end{bmatrix}$.

1.3. Consider a pair of yellow sunglasses for water sports which cuts out all blue light but passes all red and green light. Find a 3×3 matrix $\mathcal{B}$ that represents the transformation incoming light undergoes as it passes through the sunglasses.

$\mathcal{B} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 0 \end{bmatrix}$.

1.4. Is $\mathcal{B}$ invertible?

No. The rank of $\mathcal{B}$ is 2 which is less than 3.

1.5. Find the matrix for the composite transformation that light undergoes as it first passes through the sunglasses and then the eye.

This is $\mathcal{AB} = \begin{bmatrix} \frac{1}{3} & \frac{1}{3} & 0 \\ 1 & -1 & 0 \\ -\frac{1}{2} & -\frac{1}{2} & 0 \end{bmatrix}$.

1.6. As you put on the sunglasses, the signal you receive (the I, L, and S) undergoes a transformation. Find the matrix $\mathcal{M}$ of this transformation. (There is nice picture in the book if this is not clear enough.)

Lets think before begin (always a good thing). We want two compositions to be the same. In the first transformation the light goes though the sunglasses, represented by the matrix $\mathcal{B}$, and then though the eye represented. In the second transformation, light through the eye, represented by $\mathcal{A}$, and then through an unknown transformation represented by $\mathcal{M}$. So for all $\vec{x}$, we want

$\mathcal{A}\mathcal{B}\vec{x} = \mathcal{M}\mathcal{A}\vec{x}$. Hence we want $\mathcal{M}$ such that $\mathcal{A}\mathcal{B} = \mathcal{M}\mathcal{A}$. Since $\mathcal{A}$ is invertible we can multiple the right side by $\mathcal{A}^{-1}$ and hence $\mathcal{M} = \mathcal{A}\mathcal{B}\mathcal{A}^{-1}$. So we have the $\mathcal{M}$ is $\begin{bmatrix} \frac{1}{2} & 0 & -\frac{2}{9} \\ 0 & 1 & 0 \\ -1 & 0 & \frac{1}{3} \end{bmatrix}$.

2. Let B be a $m \times n$ matrix with $m \geq n$. Let A be a matrix such that BA is invertible. (Hint: look at problems 31-35 from section 2.4)

2.1. What is the size of A?

BA must be square and hence must be $n \times n$. So A is $n \times m$.

2.2. Is $T(\vec{x}) = B\vec{x}$ onto? Or equivalently, for all $\vec{b} \in \mathbb{R}^m$ there exists a vector $\vec{x} \in \mathbb{R}^n$ such that $B\vec{x} = \vec{b}$. Or equivalently, the linear system $B\vec{x} = \vec{b}$ is consistent for all vectors $\vec{b} \in \mathbb{R}^m$.

Yes, for each vector $\vec{b} \in \mathbb{R}^m$, take $\vec{x} = A\vec{b}$.

2.3. What is the rank of B?

Since the linear system $B\vec{x} = \vec{b}$ is consistent for all vectors $\vec{b} \in \mathbb{R}^m$ the rank of B must be m.

2.4. What is the realtion between n and m?

$n = m$ because $m = \text{rank}(B) \leq n \leq m$.

2.5. Apply a lemma from the book (from class) to show that B and A are invertible.

A and B are square matrices. Since the rank of B is $m = n$, $rref(B) = I_n$. Thus B is invertible. Now multiplying $BA = C$ by the inverse of B, we get $A = B^{-1}C$. Since B^{-1} and C are invertible A is invertible.

3.

3.1. Find the matrix of the linear transformation $T_{\vec{a}} : \mathbb{R}^3 \to \mathbb{R}^3$ that projects each vector $\vec{x}$ onto the line given by $\vec{a} = \begin{bmatrix} 1 \\ 0 \\ 1 \end{bmatrix}$

Call $\mathcal{A}$ the matrix of the linear transformation $T_{\vec{a}}$. I first need to find the unit vector on the line given by $\vec{a}$: $\vec{u} = \begin{bmatrix} 1/\sqrt{2} \\ 0 \\ 1/\sqrt{2} \end{bmatrix}$. $\mathcal{A} = \begin{bmatrix} 1/2 & 0 & 1/2 \\ 0 & 0 & 0 \\ 1/2 & 0 & 1/2 \end{bmatrix}$.

3.2. Find the matrix of the linear transformation $T_{\vec{b}} : \mathbb{R}^3 \to \mathbb{R}^3$ that projects each vector $\vec{x}$ onto the line given by $\vec{b} = \begin{bmatrix} 1 \\ -1 \\ 0 \end{bmatrix}$

Call $\mathcal{B}$ the matrix of the linear transformation $T_{\vec{b}}$. I first need to find the unit vector on the line given by $\vec{b}$: $\vec{v} = \begin{bmatrix} 1/\sqrt{2} \\ -1/\sqrt{2} \\ 0 \end{bmatrix}$. $\mathcal{B} = \begin{bmatrix} 1/2 & -1/2 & 0 \\ -1/2 & 1/2 & 0 \\ 0 & 0 & 0 \end{bmatrix}$.

3.3. Now use the above informations to find the matrix of the linear transformation $T : \mathbb{R}^3 \to \mathbb{R}^3$ that projects each vector $\vec{x}$ onto the plane $\mathcal{P}$ given by $\vec{a}$ and $\vec{b}$. (The plane $\mathcal{P}$ is the set of all linear combinations of $\vec{a}$ and $\vec{b}$.) (Hint: look at problem 40 from section 2.2)

Call $\mathcal{C}$ the matrix of the linear transformation $T : \mathbb{R}^3 \to \mathbb{R}^3$ that projects each vector $\vec{x}$ onto the plane $\mathcal{P}$ given by $\vec{a}$ and $\vec{b}$. $T(\vec{x}) = T_a(\vec{x}) + T_b(\vec{x}) = \mathcal{A}\vec{x} + \mathcal{B}\vec{x}$. So $\mathcal{C} = \mathcal{A} + \mathcal{B} = \begin{bmatrix} 1 & -1/2 & 1/2 \\ -1/2 & 1/2 & 0 \\ 1/2 & 0 & 1/2 \end{bmatrix}$.