Student's name:.....:

Part 1. Multiple choice (8 points each)

1. Which of the following is true for arbitrary sets A, B, C ?

Answers

a) $(A \cup B) \cap B = A \cap B$,

b) $(A \cup B) \cap B = A$,

c) $(A \cup B) \cap B = B$,

d) $(A \cup B) \cap B = A \cup B$,

e) $(A \cup B) \cap B = A' \cap B$

2. Which of the following mappings $f : \mathbf{R} \to \mathbf{R}$ is a bijection?

Answers

a) $f(x) = |x|$,

b) $f(x) = 0$,

c) $f(x) = \begin{cases} 2x & \text{if } x \text{ rational} \\ x & \text{if } x \text{ irrational} \end{cases}$

d) $f(x) = \begin{cases} x + 1 & \text{if } x < 0 \\ x - 1 & \text{if } x \geq 0 \end{cases}$

3. Which of the following binary relations on **R** is an equivalence relation?

Answers

a) $x \sim y$ if and only if $x \mid y$,

b) $x \sim y$ if and only if $x^2 = y^3$,

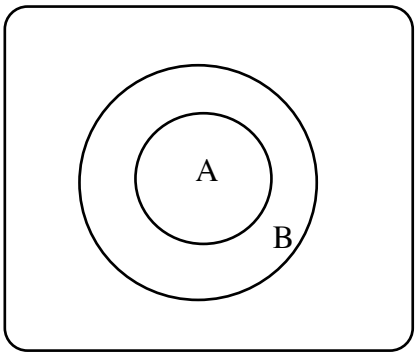
c) $x \sim y$ if and only if $x^2 = y^2$,

d) $x \sim y$ if and only if $x = 2y$,

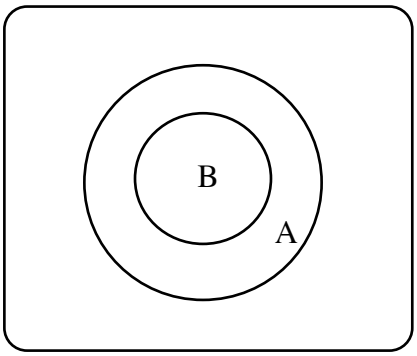
e) $x \sim y$ if and only if $xy = 1$

4. Which of the following Venn diagrams represents the relation $A \subset B'$?

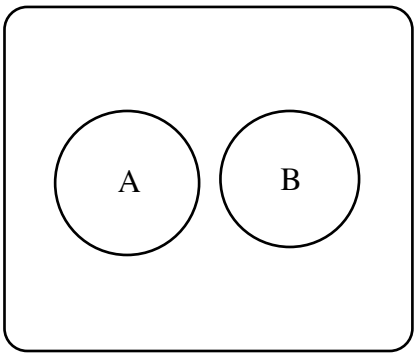
Answers



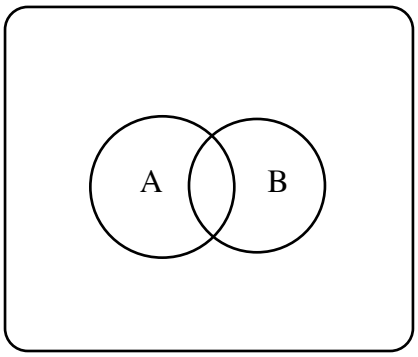
a



b



c



d

5. The last digit of the greatest common divisor of 585 and 1911 is

Answers

a) 1 b) 3 c) 5 d) 7 e) 9

6. The general solution of the congruence

$$6x \equiv 10 \pmod{8}$$

is

Answers

a) $3 + 2k$ b) $3 + 4k$ c) $3 + 6k$ d) $3 + 8k$

7. Which of the following complex numbers is a ninth root of unity?

Answers

a) $-\frac{1}{2} + \frac{\sqrt{3}}{2} i$ b) $\frac{1}{2} + \frac{\sqrt{3}}{2} i$ c) -9

d) $\frac{\sqrt{2}}{2} + \frac{\sqrt{2}}{2} i$ e) $\frac{3}{5} + \frac{4}{5} i$

8. According to the definition a primitive n-th root of unity is an n-th root of unity

a) whose real part equals zero, b) whose imaginary part is positive

c) whose real part has the smallest absolute value among all roots

d) which generates all other roots by multiplication

e) whose absolute value equals 1

9. How many distinct non-trivial proper cyclic subgroups are there in the (additive) group $\mathbf{Z}_{12}$?

Answers:

a) 2 b) 3 c) 4 d) 5 e) 6

10. Which of the following mappings is a group homomorphism $U_{12} \to \mathbf{Z}_{12}$?

Answers

a) $[1] \to [1]$, $[5] \to [5]$, $[7] \to [7]$, $[11] \to [11]$,

b) $[1] \to [0]$, $[5] \to [6]$, $[7] \to [6]$, $[11] \to [6]$,

c) $[1] \to [0]$, $[5] \to [4]$, $[7] \to [6]$, $[11] \to [10]$,

d) $[1] \to [6]$, $[5] \to [7]$, $[7] \to [9]$, $[11] \to [1]$

e) none of the above.

11. The integral domain R contains an an element a, such that $12a = 0$. Which of the following can be concluded from this fact?

Answers

a) the characteristic of the integral domain = 12,

b) the characteristic of the integral domain ≤ 12,

c) the characteristic of the integral domain ≥ 12,

d) the characteristic of the integral domain $= 0$

12. The remainder in the division in $\mathbf{Z}_3(x)$ of the polynomial $x^4 + x^3 + x + 1$ by $x^2 + 1$ is

Answers:

a) 1 b) 2 c) x d) $x + 1$ e) $x + 2$

Part 2. True or false (2 points each)

Let $*$ be a binary operation on a non-empty set A. Are the following statements true or false in general?

13. If the operation is associative and has both a left identity element and a right identity element, then they coincide.

14. If the operation has an identity element, then the operation must be associative.

True or false?

15. All seventh roots of unity are are primitive roots.

16. All ninth roots of unity are primitive roots.

Given an arbitrary complex number z, are the following statements true or false?

17. $z - \bar{z}$ is always a real number.

18. $z + \bar{z}$ is always a real number.

19. $z\,\bar{z}$ is always a real number.

20. z^2 is always a real number.

True or false?

21. Every group of order 27 is abelian.

22. Every group of order 19 is cyclic.

23. Any two groups of the same order are isomorphic.

24. Any two cyclic groups of the same order are isomorphic.

25. The subgroups of A_4 generated by the permutation (123) is a normal subgroup.

26. The subgroup of D_3 generated by the reflection f is a normal subgroup.

27. The groups S_4 and D_4 are isomorphic.

A non-abelian group G has a normal subgroup of index 3. Are the following statements true or false?

28. The factor group G/N must be a cyclic group.

29. The factor group G/N must be abelian but may not be cyclic.

True or False?

30. A group of order 25 cannot have a subgroup of order 7.

31. There exists a subgroup of order 6 and index 6 of the group $\mathbb{Z}_{144}$.

Let G be a group and N and K its subgroups. Are the following statements true or false?
($A < B$ denotes "A is a subgroup of B", $A \lhd B$ denotes "A is a normal subgroup of B)?

32. If $K < N$ and $N < G$, then $K < G$.

33. If $K < G$ and $K < N$, then $K < N$.

34. If $K \lhd G$ and $N < G$, then $NK < G$.

Consider the following subsets of the group of all 2×2 matrices over the integers. Are the following statements true or false?

35. $S = \left\{ \begin{bmatrix} x & 0 \\ 0 & y \end{bmatrix} \mid x, y \in \mathbf{Z} \right\}$ is a commutative subring.

36. $T = \left\{ \begin{bmatrix} x & y \\ 0 & 0 \end{bmatrix} \mid x, y \in \mathbf{Z} \right\}$ is a commutative subring.

37. $U = \left\{ \begin{bmatrix} 0 & x \\ x & 0 \end{bmatrix} \mid x \in \mathbf{Z} \right\}$ is a commutative subring.

Let $R = \{a + b\sqrt{2} \mid a, b \in \mathbf{Z}\}$. Is the following subring an ideal of R?

38. $S = \{a + b\sqrt{2} \mid a \in 2\mathbf{Z}, b \in \mathbf{Z}\}$,

39. $T = \{a + b\sqrt{2} \mid a \in \mathbf{Z}, b = 0\}$