Math 222, Section 2 Spring 1991

REVIEW SHEET FOR TEST NO. 3

1. You should be familiar and comfortable with the concepts of a ring, commutative ring, zero and unity of a ring, proper zero divisors, units, multiplicative inverse, integral domain, field, subrings, isomorphism of rings.

   You should be able to determine whether a given set with two operations is a ring and what particular type of a ring it is. You should be able to determine whether a subset of a ring is is a subring. Theorem 5.4 (p. 160) is especially convenient for this purpose. You should also be aware that (and why) in the case of a finite ring closure under addition and multiplication suffices. Note that universal properties like being commutative, distributive, having no proper zero divisors are inherited by subset, while existential properties like having a unity or having an inverse element are not.

   You should be familiar with examples of rings: $\mathbf{Z}$, $\mathbf{Z}_n$ and their subrings, $\mathbf{Q}$, $\mathbf{R}$, $\mathbf{C}$, the ring of matrices and some of their subrings, direct sums of rings, extensions of $\mathbf{Z}$ and $\mathbf{Q}$ like the ones discussed in exercises 5.1 problems 1 and 2.

Exercises:

1.1. Prove based only on the definition of a ring R that for every $x, y, a, b \in R$

$$(x + y)(a + b) = xa + xb + ya + yb.$$

1.2. Prove that a ring has no proper zero divisors if and only if it satisfies the two cancellation laws: for $a \neq 0$ and any x and y

$$ax = ay \text{ implies } x = y \quad \text{and} \quad xa = ya \text{ implies } x = y.$$

1.3. Do problem 24 on p. 168. This gives you an example of a subring $R = \{[0], [2], [6], [8]\}$ of the ring $\mathbf{Z}_{10}$. The ring R does not contain the unity of the ring $\mathbf{Z}_{10}$ but it has a unity of its own, namely [6], which in turn is not the unity of the larger ring. Each nonzero element has an inverse in R, e.g. the inverse of [2] in the ring R is [8], since $[6]\cdot[8] = [6]$ — the unity of R. However, in the ring $\mathbf{Z}_{10}$, [2] is a zero divisor, namely $[2]\cdot[5] = [0]$, and has no inverse.

1.4. Prove that a subring of a field is an integral domain but not necessarily a field.

1.5. In the following table fill in the information about the sets with operations given in the first column. Write "yes" or "no" in columns 2, 3, 6, 7, and write (if any) the unity an a pair of zero divisors in columns 4 and 6 respectively. If these object do not exist write "no".

1 Set and operations			4 Unity	5 Zero Diviso		
$\{z \in \mathbf{C} \mid \lvert z \rvert = 1\}$ with usual operations in $\mathbf{C}$						
$\{yi \in \mathbf{C} \mid y \in \mathbf{R}\}$ with usual operations in $\mathbf{C}$						
$\{m + n\sqrt{8}\ i \mid m, n \in \mathbf{Z}\}$ with us operations in $\mathbf{C}$						
$\left\{\begin{pmatrix} x & y \\ 0 & 0 \end{pmatrix} \mid x, y \in \mathbf{Z}\right\}$ with mat operations						
$\left\{\begin{pmatrix} x & 0 \\ 0 & y \end{pmatrix} \mid x, y \in \mathbf{Z}\right\}$ with ma operations						
$\left\{\begin{pmatrix} x & x \\ 0 & 0 \end{pmatrix} \mid x \in \mathbf{Z}\right\}$ with matrix operations						
$\left\{\begin{pmatrix} x & x \\ 0 & 0 \end{pmatrix} \mid x \in \mathbf{Q}\right\}$ with matrix operations						
$\mathbf{Z}_7$						

$\mathbf{Z}_3 \oplus \mathbf{Z}_5$						

2. You should be familiar and comfortable with the concepts of a left ideal, right ideal, ideal of a ring and principal ideal of a commutative ring and be able to tell whether a given subset of a ring is an ideal (principal ideal). You should know and be able to prove that every subring of the rings $\mathbf{Z}$ and $\mathbf{Z}_n$ is a principal ideal (see Theorem 6.3 on p. 193).

You should be familiar and comfortable with the concepts of of a quotient ring *R*/*I*, ring homomorphism, epimorphism and isomorphism, kernel of a homomorphism and know the Fundamental Theorem of Ring Homomorphisms (theorems 6.12, 6.13 and its corollary, pp. 203-204).

You should know the concept of characteristic of a ring and be able to find it for familiar rings. You should know and be able to prove that the characteristic of a ring with unity equal the additive order of the unity, if the order is finite, and 0, if the order is infinite (see Theorem 6.16, p. 208). You should know the assertions and the proof of Theorem 6.17 and 6.18 (pp. 208-209).

Exercises

2.1 Prove that a non empty subset I of a ring R is an ideal if and only if $x - y$, rx and xy belong to I for all $x, y \in I$, $r \in R$. (Exercise 1 on p. 196) This theorem provides a very convenient way to check if a subset is an ideal.

2.2 List all nontrivial ideals of a) $\mathbf{Z}_3$, b) $\mathbf{Z}_9$, c) $\mathbf{Z}_{12}$, d) $\mathbf{Z}_{30}$. Since here every ideal is a principal ideal write every ideal in the form (a) — as a principal ideal generated by a.

2.3 Consider the ring $S = \left\{ \begin{pmatrix} x & y \\ 0 & z \end{pmatrix} \mid x, y, z \in \mathbf{Z} \right\}$ with matrix operations.
Decide which of the following area left ideals, right ideals or ideals of this ring S.

a) $\left\{ \begin{pmatrix} a & b \\ 0 & 0 \end{pmatrix} \mid a, b \in \mathbf{Z} \right\}$, b) $\left\{ \begin{pmatrix} a & 0 \\ b & 0 \end{pmatrix} \mid a, b \in \mathbf{Z} \right\}$,

c) $\left\{ \begin{pmatrix} a & 0 \\ 0 & b \end{pmatrix} \mid a, b \in \mathbf{Z} \right\}$, d) $\left\{ \begin{pmatrix} a & 0 \\ 0 & 0 \end{pmatrix} \mid a \in \mathbf{Z} \right\}$,

e) $\left\{ \begin{pmatrix} 0 & 0 \\ 0 & a \end{pmatrix} \mid a \in \mathbf{Z} \right\}$ f) $\left\{ \begin{pmatrix} 0 & a \\ 0 & 0 \end{pmatrix} \mid a \in \mathbf{Z} \right\}$.

2.4 Which of the following mapping from $\mathbf{C}$ to $\mathbf{R}$ is a homomorphism of rings?

a) $z \to \operatorname{Re} z$, b) $z \to \operatorname{Im} z$ c) $z \to |z|$, d) $z \to \bar{z}$.
In the case it is a homomorphism what is its kernel?

2.5. Consider the ring $S = \{a + b\sqrt{2} \mid a, b \in \mathbf{Z}\}$. Which of the following mappings $\varphi: S \to S$ is a ring homomorphism? What is its kernel?

a) $\varphi(a + b\sqrt{2}) = a - b\sqrt{2}$,

b) $\varphi(a + b\sqrt{2}) = a$,

c) $\varphi(a + b\sqrt{2}) = b$

d) $\varphi(a + b\sqrt{2}) = ab$.

2.6 What is the characteristic of the following rings?

a) $\mathbf{Z}$,

b) $\mathbf{Z}_7$,

c) $\mathbf{Z}_{24}$,

d) $\mathbf{Z}_3 \oplus \mathbf{Z}_5$,

e) $\mathbf{Z}_6 \oplus \mathbf{Z}_{15}$,

f) $\left\{ \begin{pmatrix} x & 0 \\ 0 & y \end{pmatrix} \mid x \in \mathbf{Z}_4, y \in \mathbf{Z}_6 \right\}$.

3. You should be skilful in computations with complex numbers, in particular finding n-th roots of complex numbers.