1. (15) Describe the sets represented by the following Venn diagrams.

a)

b)

c)

2. (20) For each of the mappings

$f : \mathbb{Z} \to \mathbb{Z}$ given below,

determine if the mapping is surjective, injective, or bijective. Justify your answers.

a)(5) $f_1(x) = \begin{cases} 1 & , \text{ if } x \text{ is odd} \\ 5x & , \text{ if } x \text{ is even} \end{cases}$

b)(5) $f_2(x) = 5x + 1$

c)(10) Compute $(f_2 \circ f_1)(x)$ and determine if it is surjective, injective, or bijective.

3. (10) Find the greatest common divisor and integers m and n such that

$$(a,b) = am + bn$$

1) $a = 99$, $b = 204$

2) $a = -10$, $b = 66$

4. (20) Find a solution $x \in \mathbb{Z}$, $0 \le x < n$, for the following congruences

a) $5x = 7 \pmod 9$

b) $12x \equiv 16 \pmod{24}$

5. (30) Consider the following subset of the ring M_{2x2} of 2x2 matrices over $\mathbb{Z}$

$$S = \left\{ \begin{bmatrix} a & b \\ 0 & c \end{bmatrix} \middle| \; a, b, c \in \mathbb{Z} \right\}$$

a) Show that S is a subring of M_{2x2} with respect to matrix addition and multiplication

b) Show that

$$I = \left\{ \begin{bmatrix} 0 & d \\ 0 & 0 \end{bmatrix} \middle| \; d \in \mathbb{Z} \right\}$$

is an ideal of S.

c) Show that the mapping

$\Phi : S \to \hat{I}$ defined by

$\Phi \left(\begin{bmatrix} a & b \\ o & c \end{bmatrix} \right) = a^2$

is an epiomorphism

d) Describe ker Φ

e) Describe M/I = , quotient ring of I.

6. (10) Let $\mathbb{R}$ be a field of real numbers. List all ideals of $\mathbb{R}$. Justify your answer.

7. (15) Let $G = \langle a \rangle$ be a cyclic group of order 15.

a) List all generators of G

b) List all distinct subgroups of G

c) What is the cyclic subgroup of $\mathbb{Z}$ generated by (–2) under + ?

8. (10) In $(\mathbb{Z}_6, +)$, write down all the cosets of the subgroup $H = \{[0], [3]\}$. What is the index of H?

9. (10) Show that for x, y, z $\in$ D, an ordered integral domain, the following is true

a) If $x > y$ and $y > z$, then $x > z$

b) If $x > y$, then $2x > 2y$

10. (10) Find the characteristic of the following rings. Justify your answer.

a) $\mathbb{Z}_{21}$

b) $\mathbb{Z}_3 \otimes \mathbb{Z}_4$