Math 222 Exam 2: Friday, March 21, 1997; 1:55–2:45pm

Instructions. Use your time wisely; the four questions are not of equal value. Calculators may be used if desired. Throughout the exam, $\mathbf{Z}$ denotes the integers and $\mathbf{Z}_m$ denotes the integers modulo m.

1. *(30 points)*

(a) Express the following complex numbers in the form $a + bi$ for real numbers a, b;
(i) $(2+i)-(3-2i)$ (ii) $(2+i)(3-2i)$ (iii) $\overline{3-2i}$ (iv) $|3-2i|$ (v) $(3+2i)^{-1}$.

(b) If the polar form of a complex number z is $z = r(\cos\theta + i\sin\theta)$ and m is a positive integer, express z^m in polar form. Find all the complex numbers z such that $z^4 = -4$ and express each of them in the form $a + bi$ with a, b real numbers.

(c) Calculate the following quaternions, expressing the answers in the form $a + bi + cj + dk$ with a, b, c, d real numbers;
(i) $(2+i-j+3k)+(3+2i-2j-2k)$ (ii) $(1+i-j)(i+j+k)$ (iii) $(2-i+2j-2k)^{-1}$.

2. *(30 points)* Let R denote the ring of polynomials $\mathbf{Z}_3[x]$ in the variable x over $\mathbf{Z}_3$ and F denote the field of fractions of R.

(a) Let $f = \overline{2} + x + \overline{2}x^2$ and $g = \overline{2} + x$ Express the polynomials (i) $f - g^2$ and (ii) g^3 in the form $a_0 + a_1x + \ldots + a_nx^n$ with each $a_i \in \mathbf{Z}_3$ one of $\overline{0}$, $\overline{1}$ or $\overline{2}$.

(b) In F, express $x^{-1} + (x+\overline{1})^{-1} + (x+\overline{2})^{-1}$ in the form $\frac{h}{k}$ with h, $k \in R$.

(c) Prove that for any $h \in R$, $h + h + h = 0$ (hint; first explain why $h + h + h = (\overline{1}+\overline{1}+\overline{1})h$). Is $h + h + h = 0$ true for any $h \in F$? Give reasons for your answer.

(d) Let S denote the subring of R generated by $\mathbf{Z}_3$ and $x^2 + \overline{1}$. Which of the following elements of R are in S
(i) x^2 (ii) $x^4 + \overline{2}x^2 + \overline{2}$ (iii) $x^2 + x + \overline{2}$?
Under what conditions on $a_0, a_1, \ldots, a_n \in \mathbf{Z}_3$ is $a_0 + a_1x + \ldots + a_nx^n$ an element of S?

3. *(25 points)*

(a) Prove carefully that the relation $\sim$ on $\mathbf{Z}$ defined by $a \sim b$ if $|a| = |b|$, where $|\ |$ denotes absolute value, is an equivalence relation. List the elements of each of the equivalence classes $[0]$, $[1]$ and $[2]$.

(b) Let $Y = \mathbf{Z}/\sim$ denote the set of equivalence classes for the equivalence relation in (b). Prove that there is a well-defined function $f\colon Y \to \mathbf{Z}$ such that $f([x]) = x^2$ for any $x \in \mathbf{Z}$, but there is no function $g\colon Y \to \mathbf{Z}$ with $g([x]) = x^3$. Calculate $f([0])$ and $f([2])$. For which positive integers m is there a well-defined function $h\colon Y \to \mathbf{Z}$ with $h([x]) = x^m$?

4 *(15 points)* Let R be a commutative ring

(a) Prove from the ring axioms that if a, b, c, d are elements of R, then

$$(a^2 + b^2)(c^2 + d^2) = (ac - bd)^2 + (ad + bc)^2.$$

(b) An element m of R is said to be a sum of two squares if there exist elements a and b of R with $a^2 + b^2 = m$. (For instance, in $\mathbf{Z}$, $10 = 3^2 + 1^2$ and $9 = 3^2 + 0^2$ are sums of two squares but 3 is not). Use the equation in (a) to prove that if elements m and n of R are both sums of two squares, then mn is also a sum of two squares.

(c) Assume that R is the field of real numbers. Express the identity in (a) in terms of the absolute values of the complex numbers $z = a + bi$, $w = c + di$ and zw.