Math 222 Exam 3: Wednesday, April 16, 1997; 1:55–2:45pm

Instructions. Use your time wisely; the four questions are not of equal value. Calculators may be used if desired. Throughout the exam, $\mathbf{Z}$ denotes the integers, $\mathbf{Q}$ denotes the rational numbers and $\mathbf{Z}_m$ denotes the integers modulo m.

1. *(25 points)* Let $F[X]$ denote the polynomial ring over a field F.

(a) Explain carefully what is meant by saying that a polynomial $f \in F[X]$ is irreducible.

(b) State in detail the theorem on unique factorization in $f[X]$.

(c) What is meant by saying that an element α of F is a root of f? What is the relationship between roots α of f and linear factors $X - \beta$, $\beta \in F$, of f?

(d) Explain why a polynomial $g \in F[X]$ of degree n with no roots in F is irreducible if $n = 2$ or $n = 3$, but is not necessarily irreducible if $n = 4$.

2. *(30 points)*

(a) State the theorem on rational roots of a polynomial $a_0 + a_1X + \ldots + a_nX^n$ with integral coeefficients.

(b) Use the theorem in (a) to find all the roots in $\mathbf{Q}$ of the polynomial $2X^4 - X^3 + X^2 - X - 1$.

(c) Carefully state Eisenstein's criterion for irreducibility of a polynomial $f = a_0 + a_1X + \ldots + a_nX^n \in \mathbf{Q}[X]$.

(d) Show using Eisenstein's criterion that the polynomial $x^6 - 12 \in \mathbf{Q}[X]$ is irreducible, and conclude that $\sqrt[6]{12}$ is an irrational number.

(e) Decide whether the polynomials (i) $X^4 + 4 \in \mathbf{Q}[X]$ and (ii) $X^2 + X + 2 \in \mathbf{Z}_3[X]$ are irreducible.

3. *(20 points)* Let $\alpha = \sqrt[6]{12}$ be the positive real root of the irreducible polynomial $f = X^6 - 12 \in \mathbf{Q}[X]$, and $\mathbf{Q}[\alpha]$ be the subring of $\mathbf{R}$ generated by $\mathbf{Q}$ and α.

(a) Use Euclid's algorithm to express the greatest common divisor d of f and $g = X^2 - 2$ in the form $d = af + bg$ for suitable a, b in $\mathbf{Q}[X]$.

(b) Use your answer to (a) to express $(\alpha^2 - 2)^{-1}$ in the standard form $a_0 + a_1\alpha + \ldots + a_5\alpha^5$ with the $a_i \in \mathbf{Q}$.

(c) Express $(\alpha^4 + 2)^2 \in \mathbf{Q}[\alpha]$ in the standard form in (b).

(d) Is $\mathbf{Q}[\alpha]$ a field? Justify your answer.

4. *(25 points)* Consider the quotient ring $S = R/I$ where R is the polynomial ring $R = \mathbf{Z}_2[X]$ and I is the principal ideal $I = \langle X^2 + 1 \rangle$ of R. You may use the fact that $S = \{\overline{0}, \overline{1}, \overline{X}, \overline{X+1}\}$ has four elements.

(a) Write out the multiplication and addition tables for S.

(b) Determine whether S is an integral domain or a field.

(c) Explain what is meant by saying that a subset J of S is an ideal. Find all the ideals of S (there are three of them including S and the zero ideal). Is every ideal of S a principal ideal? Is every ideal of R principal?