

Name: \_\_\_\_\_

November 17, 2004

**Exam 1**

This Examination contains 7 problems on 6 sheets of paper Do all your work in this booklet and show your computations. Calculators, books, and notes are not allowed.

**Points**

<u>Question</u>	<u>Possible</u>	<u>Earned</u>	<u>Question</u>	<u>Possible</u>	<u>Earned</u>
1	10		5	15	
2	15		6	15	
3	15		7	15	
4	15				
			Total	100	

1. Let  $\alpha = -3 + 7i \in \mathbb{Z}[i]$ . Find another number  $\beta \in \mathbb{Z}[i]$  so that  $\alpha\beta = m(1 + i)$  for some integer  $m \in \mathbb{Z}$ .

2.

(a). Use the division algorithm for integers to find the base 5 representation of the number 303. In other words, find coefficients

$$0 \leq r_i < 5$$

so that

$$303 = r_m 5^m + r_{m-1} 5^{m-1} + \cdots + r_1 5 + r_0.$$

(b). What would you consider a reasonable base 5 representation for  $-1515$ ?

3. Consider the ring  $\mathbb{Z}[\sqrt{5}] = \{a + b\sqrt{5} : a, b \in \mathbb{Z}\}$ .

(a). Find an integer  $a$  so that the number  $\alpha = a + \sqrt{5}$  satisfies the inequality  $0 < \alpha < 1$ .

(b). Show that there exists a number  $\beta \in \mathbb{Z}[\sqrt{5}]$  which lies between 2 and  $\frac{9}{4}$  on the number line.

(c). Assume that you know  $\sqrt{5}$  to be an irrational number. Let  $\alpha = a + b\sqrt{5} \in \mathbb{Q}(\sqrt{5})$ , so  $a, b \in \mathbb{Q}$ . Suppose  $\alpha \neq 0$ . Find a formula for  $c, d \in \mathbb{Q}$  so that

$$\frac{1}{\alpha} = c + d\sqrt{5}.$$

Why do you need to know that  $\sqrt{5}$  is irrational? Explain this in detail.

4. For each of the pairs  $\alpha, \beta \in \mathbb{Z}[i]$  below, find  $\gamma, \rho \in \mathbb{Z}[i]$  so that

$$\beta = \alpha\gamma + \rho, \quad \text{and} \quad N(\rho) < N(\alpha).$$

(a).  $\alpha = -1 + 3i, \beta = 21$

(b).  $\alpha = 1 - 1i, \beta = 2 + 8i$

5. For each pair of polynomials  $f$  and  $g$  below, find polynomials  $q$  and  $r$  so that

$$g = qf + r \quad \text{and} \quad \deg(r) < \deg(f).$$

In each case, regard the coefficients of  $f$  and  $g$  as coming from the coefficient ring  $R$ .

(a).  $R = \mathbb{Z}[\sqrt{2}]$ ,  $g = X^3 - X^2 + X + 1$ ,  $f = X - (1 + \sqrt{2})$ .

(b).  $R = \mathbb{Z}_{13}$ ,  $g = X^4 + 12$ ,  $f = X^3 + X^2 + X + 1$ .

6. Suppose that  $y$  is a root of the polynomial

$$X^5 + (n - 1)$$

in  $\mathbb{Z}_n$ , where  $n$  is a positive integer. Show that  $y$  is a unit in  $\mathbb{Z}_n$ .

7. Let  $a = 90$  and  $b = 168$ .

(a). Find  $\gcd(a, b)$ .

(b). Find  $t, s \in \mathbb{Z}$  so that  $ta + sb$  divides both  $a$  and  $b$ .