Name: _____

Exam 2

This examination contains 6 problems on 6 sheets of paper. Show all your work. Calculators, books, and notes are not allowed.

POINTS

Question	Possible	Earned	Question	Possible	Earned
1	16		4	17	
2	16		5	17	
3	17		6	17	
			Total	100	

1. Consider the ring $\mathbb{Z}_{20}$.

(a). Find all the units in this ring.

(b). Find a number between 1 and 20 which represents the quotient $\frac{3}{11}$ in $\mathbb{Z}_{20}$.

2.

(a). Find all greatest common divisors of $1-i$ and $2-13i$ in $\mathbb{Z}[i]$ and give a Bezout-equation for one of them.

(b). Write $3 = \sigma(1-i) + \tau(2-13i)$ for some choice of $\sigma, \tau \in \mathbb{Z}[i]$.

3. Use the rational roots test to do the following problems.

(a). Let p be a prime number. Show that the polynomial

$$f(X) = X^4 + p^3X + p$$

has no roots in $\mathbb{Q}$.

(b). Show that $\sqrt[98]{19}$ is irrational.

4. Let p be a prime number, and suppose that u is a unit of order 3 in $\mathbb{Z}_p$.

(a). Show that $p \equiv 1 \pmod 6$.

(b). Show that u satisfies a monic polynomial equation of degree 3 of the form $X^3 - a = 0$ for some choice of $a \in \mathbb{Z}_p$. What is a?

(c). Use (b) to prove that there are precisely 2 units of order 3 in $\mathbb{Z}_p$. What are they?

5. Consider the ring $\mathbb{Z}_{41}$.

(a). Assume you know that 5 has order 20 in $\mathbb{Z}_{41}$. What is the order of 5^3? Find $k \in \mathbb{Z}$ so that 5^k has order 5.

(b). You are given that $13^2 \equiv 5 \pmod{41}$. What is the order of 13 in $\mathbb{Z}_{41}$? (Be sure to justify your answer. Of course, you may use what you were told in part (a).)

6. Let R be a commutative ring. Let $a \in R$ be a unit of order 14, let $b \in R$ be a unit of order 22, and let $c \in R$ be a unit of order 10. Find the largest possible value of the order of abc. Give all details; don't just tell me the numerical answer. You might try first estimating the order of ab, and then of $(ab)c$.