Name: ______________________

Exam 3

This examination contains 4 problems on 5 sheets of paper. Show all your work. I doubt that calculators will help much.

POINTS

Question	Possible	Earned	Question	Possible	Earned
1	25		3	25	
2	25		4	25	
			Total	100	

1. Consider the group S_7, the symmetric group on $\{1,2,\dots,7\}$. Let

$$\sigma = \begin{pmatrix} 1 & 2 & 3 & 4 & 5 & 6 & 7 \\ 7 & 3 & 2 & 1 & 4 & 5 & 6 \end{pmatrix} \in S_7$$

(a). Write σ in cycle notation.

(b). Compute the order of σ.

(c). Decide whether σ lies in A_7, the alternating group. Be sure to explain carefully your reasoning.

2. You are given that the group of units of $\mathbb{Z}_{11}$ is cyclic and is generated by, for example, the number 6 which therefore has order 10 (you may check this fact if you like, but this is not required.)

Recall we showed in class that $\mathbb{Z}_{11}[i]$ is a field, where $i^2 = -1 = 10$. There are $11^2 - 1 = 120$ units in $Z_{11}[i]$.

(a). Let $\alpha = 4 + 3i \in \mathbb{Z}_{11}[i]$. Show that α has order 60 in $\mathbb{Z}_{11}[i]$. (Hint: what is the order of α^6?)

(b). Find a solution β in $\mathbb{Z}_{11}[i]$ to the equation $X^2 = \alpha$. Deduce that β has order 120 and hence that the unit group of $\mathbb{Z}_{11}[i]$ is cyclic. (See hint below.)

(Hint: if $\beta = a + bi$, then $\beta^2 = (a^2 - b^2) + 2abi$, so you need to find $a, b \in \mathbb{Z}_{11}$ so that $a^2 - b^2 = 4$ and $2ab = 3$. The following table may be helpful – you fill in the a^2 column.

b	$a = \frac{3}{2b}$	b^2	a^2	b	$a = \frac{3}{2b}$	b^2	a^2
1	7	1		6	3	3	
2	9	4		7	1	5	
3	6	9		8	5	9	
4	10	5		9	2	4	
5	8	3		10	4	1	

Now find β.)

(c). Find a unit $u \in \mathbb{Z}_{11}[i]$ or order 15. (You may leave your representation of u in any form you like).

3. Consider the d cycle $\sigma_d = (1,2,3,4,\ldots,d)$ in S_n for $1 \le d \le n$. In particular, $\sigma_1 = 1$, the identity permutation.

(a). Fix $2 \le d \le n$. Find a transposition $\tau = (i, j) \in S_n$ so that $\sigma_d \tau = \sigma_{d-1}$.

(b). Using (a), prove by induction that $\mathrm{sign}(\sigma_d) = (-1)^{d-1}$.

(c). What is the condition on d so that $\sigma_d \in A_n$, the alternating group?

4. Consider the group D_8, the group of symmetries of a 8-gon. Let $r = (123\dots8)$ be the clockwise rotation of the 8-gon; r generates the group $R_8 < D_8$.

Let $f_1 = (28)(37)(46)$ denote the "flip" which leaves the vertices 1 and 5 of the 8-gon fixed.

(a). Prove that $r^4 f_1 = f_1 r^4$.

(b). Let $H = \{1, r^4, f_1, r^4 f_1\}$. Prove that H is a subgroup of G (of order 4, of course).

(c). Prove that H is not a cyclic group, but that H *is* Abelian (i.e commutative).