Name: ________________________

Final

This examination contains 5 problems (and one EXTRA CREDIT problem) on 6 sheets of paper. Show all your work. I doubt that calculators will help much. Let me stress that ignoring the extra credit won't hurt you! Your score will be regarded as being from 100 points.

POINTS

Question	Possible	Earned	Question	Possible	Earned
1	20		4	20	
2	20		5	20	
3	20		Extra	5	
			Total	100	

1. Let $p = 151$. It is a fact that p is a prime number. Show that each element of $\mathbb{Z}_p^\times$ has a 7th root. (Hint: consider the function $f : \mathbb{Z}_p^\times \to \mathbb{Z}_p^\times$ given by $f(x) = x^7$. Argue that f is *onto*.)

2. For the following pairs of numbers, answer the given questions. Be sure to pay attention to which ring you are to consider the numbers in.

(a). Find the greatest commond divisor of $a = 980, b = 3920 \in \mathbb{Z}$.

(b). Let $\alpha = 5 + 5i, \beta = 7 - i \in \mathbb{Z}[i]$. Factor α and β into primes in $\mathbb{Z}[i]$, and then find their greatest common divisor.

3. Let $\alpha = a + b\sqrt{-3} \in \mathbb{Q}(\sqrt{-3})$, where $a, b \in \mathbb{Q}$ and put

$$M_\alpha = \begin{bmatrix} a & -3b \\ b & a \end{bmatrix}.$$

Let

$$F_\alpha(T) = \det\left(\begin{bmatrix} T & 0 \\ 0 & T \end{bmatrix} - M_\alpha\right)$$

denote the *characteristic polynomial* of M_α.

(a). Show that α and $\overline{\alpha}$ are roots of $F_\alpha(T)$. (Hint: Use the formula $F_\alpha(\alpha) = \det\left(\begin{bmatrix} \alpha & 0 \\ 0 & \alpha \end{bmatrix} - M_\alpha\right)$ and a similar formula for $F_\alpha(\overline{\alpha})$.)

(b). If $\alpha + \overline{\alpha} \in \mathbb{Z}$ and $N(\alpha) \in \mathbb{Z}$, show that the polynomial $F_\alpha(T)$ has *integer coefficients*. (Thus, α and $\overline{\alpha}$ are *algebraic integers*.)

(c). Let $\alpha = \dfrac{1+\sqrt{-3}}{2} \in \mathbb{Q}(\sqrt{-3})$. Use (b) to show that α is an algebraic integer (despite the denominator).

4. Let $G = D_6$ be the group of symmetries of a regular hexagon. Label the vertices of the hexagon with the numbers $1, 2, \ldots, 6$, and regard the elements of the group G as permutations of these 6 vertices.

Let H denote the set of all elements $\sigma \in G$ so that σ permutes the vertices $S = \{1, 3, 5\}$. More formally,

$$H = \{\sigma \in G : \sigma(1) \in S,\ \sigma(3) \in S,\ \text{and } \sigma(5) \in S\}.$$

(a). Show that H is a subgroup.

(b). Find the order $|H|$ of H.

(c). Let $\tau = (14)(23)(56) \in G$. Show that each element γ in the coset τH satisfies $\gamma(1) \in \{2, 4, 6\}$.

5. Let S_n denote the symmetric group on the set $\{1,2,\ldots,n\}$, and let A_n denote the alternating group (the subgroup of even permutations).

(a). Show for $n > 3$ that A_n contains a permutation of order 2. (Let me remind you that *transpositions* are odd permutations and hence aren't in A_n.)

(b). Is the conclusion of (a) true when $n = 3$? Hint: what is $|A_3|$?

(c). Show for $n > 3$ that A_n is non-abelian (i.e. not commutative.)

EXTRA CREDIT: (5 points) Let $\omega = 2+\sqrt{3} \in \mathbb{Z}[\sqrt{3}]$. Write

$$\omega^k = a(k) + b(k)\sqrt{3}$$

for $k \geq 0$, where $a(k), b(k) \in \mathbb{Z}$. Show that

$$\lim_{k\to\infty} \frac{a(k)}{b(k)} = \sqrt{3}.$$