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Mathematics 222.02: Algebraic Structures
Spring Semester 1998
Exam 1
February 9, 1998

This Examination contains 7 problems on 6 sheets of paper including the front cover. Do all your work in this booklet and show your computations. Calculators, books, and notes are not allowed.

Scores

Question	Possible	Actual
1	10	
2	15	
3	15	
4	15	
5	15	
6	15	
7	15	
Total	100	

Sign the pledge:

"On my honor, I have neither given nor received unauthorized aid on this Exam."

Signature: ______________________________

GOOD LUCK

1. Let $\alpha = -3 + 7i \in \mathbb{Z}[i]$. Find another number $\beta \in \mathbb{Z}[i]$ so that $\alpha\beta = m(1+i)$ for some integer $m \in \mathbb{Z}$.

2.

(a) Use the division algorithm for integers to find the base 5 representation of the number 303. In other words, find coefficients

$$0 \leq r_i < 5$$

so that

$$303 = r_m 5^m + r_{m-1} 5^{m-1} + \cdots + r_1 5 + r_0.$$

(b) What would you find a reasonable base 5 representation for -1515?

3. Consider the ring $\mathbb{Z}[\sqrt{5}] = \{a + b\sqrt{5} : a, b \in \mathbb{Z}\}$.

(a) Find an integer a so that the number $\alpha = a + \sqrt{5}$ satisfies the inequality $0 < \alpha < 1$. Show also that α is a unit in $\mathbb{Z}[\sqrt{5}]$.

(b) Show that there exists a number $\beta \in \mathbb{Z}[\sqrt{5}]$ which lies between 1 and $\frac{5}{4}$ on the number line.

(c) Assume that you know $\sqrt{5}$ to be an irrational number. Let $\alpha = a + b\sqrt{5} \in \mathbb{Q}(\sqrt{5})$, so $a, b \in \mathbb{Q}$. Suppose $\alpha \neq 0$. Find a formula for $c, d \in \mathbb{Q}$ so that

$$\frac{1}{\alpha} = c + d\sqrt{5}.$$

Where do you need $\sqrt{5}$ being irrational? Explain this in detail.

4. For each of the pairs $\alpha, \beta \in \mathbb{Z}[i]$ below, find $\gamma, \rho \in \mathbb{Z}[i]$ so that

$$\beta = \alpha\gamma + \rho, \quad \text{and} \quad N(\rho) < N(\alpha).$$

(a) $\alpha = -1 + 3i$, $\beta = 21$

(b) $\alpha = 1 - 1i$, $\beta = 2 + 8i$

5. For each pair of polynomials f and g below, find polynomials q and r so that

$$g = qf + r \quad \text{and} \quad \deg(r) < \deg(f).$$

In each case, regard the coefficients of f and g as coming from the coefficient ring R.

(a) $R = \mathbb{Z}[\sqrt{2}], \quad g = X^3 - X^2 + X + 1, \quad f = X - (1 + \sqrt{2}).$

(b) $R = \mathbb{Z}_{13}, \quad g = X^4 + 12, \quad f = X^3 + X^2 + X + 1.$

6. Suppose that y is a root of the polynomial

$$X^5 + (n - 1)$$

in $\mathbb{Z}_n$, where n is a positive integer. Show that y is a unit in $\mathbb{Z}_n$.

7. Let $a = 90$ and $b = 168$.

(a) Find $\gcd(a, b)$.

(b) Find $t, s \in \mathbb{Z}$ so that $ta + sb$ divides both a and b.