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Mathematics 222.02: Algebraic Structures
Spring Semester 1998
Final Exam
May 6, 1998

This Examination contains 14 problems on 7 sheets of paper including the front cover. Do all your work in this booklet and show your computations. Calculators, books, and notes are not allowed.

Question	Possible	Actual
1	10	
2	15	
3	10	
4	10	
5	10	
6	5	
7	15	

Question	Possible	Actual
8	5	
9	15	
10	10	
11	10	
12	10	
13	10	
14	15	
Total	150	

Sign the pledge:

"On my honor, I have neither given nor received unauthorized aid on this Exam."

Signature: ______________________________

GOOD LUCK

1. Find the set of all Gaussian integers which are multiples of i.

2. **(a)** Give all roots of the polynomial $f = X^2 + X + 2$ in $\mathbb{Z}_6$.

(b) Give all roots of the polynomial $f = X^2 + 4X + 3$ in $\mathbb{Z}_8$. Factorize f over $\mathbb{Z}_8$ as much as possible. [Check your answer!]

3. Consider the additive group $\mathbb{Z}_{18}$. Give all different cosets of the subgroup generated by the element 3 in $\mathbb{Z}_{18}$.

4. Given the polynomials $f = 2X^4 + 10X^3 + 2X + 5$ and $g = X^2 + 3X + 4$ with coefficients in $\mathbb{Z}_{12}$. Find polynomials a and r with coefficients in $\mathbb{Z}_{12}$ so that $f = ag + r$ and $\deg r < \deg g$.

5. Find all roots of the polynomial $p = X^4 + 4X^3 + 6X^2 + 5X + 2$ in $\mathbb{Q}$.

6. Find $\dfrac{6}{7}$ in $\mathbb{Z}_{12}$. Express the result as a number between 0 and 11.

7. Consider the multiplicative group $\mathbb{Z}_{61} \backslash \{0\}$.

(a) Show that this group contains no element of order 9.

(b) Calculate the order of 3.

(c) Give an element of order 5.

8. Find a solution of the equation $7X \equiv 3 \pmod{12}$.

9. **(a)** Define carefully the notion of a zero divisor.

(b) Let a and n be nonzero integers. Show: If a and n are not relatively prime, then a is a zero divisor in $\mathbb{Z}_n$.

10. Prove by induction:

$$1^3 + 2^3 + \ldots + n^3 = \left(\frac{n(n+1)}{2}\right)^2 \qquad \text{for all } n \geq 1$$

11. **(a)** Factorize 29 into Gaussian primes.

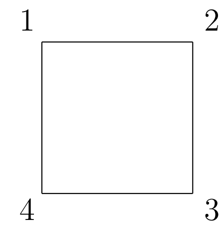
(b) Use a) to find a greatest common divisor of 29 and $1 + 12i$ in $\mathbb{Z}[i]$.

12. Show that if σ, $\tau \in S_n$, then $\sigma\tau^2\sigma \in A_n$.

13. Show that if G is a group of prime order $p = |G|$, then G is a cyclic group.

14. Consider the group D_4 of all symmetries of the square. Remember D_4 is a subgroup of S_4.

(a) List all elements of $D_4 \cap A_4$; use the picture for the notation of the vertices.



(b) Find the smallest subgroup of D_4 which contains $(1\,2\,3\,4)$ and $(1\,4)(2\,3)$.

(c) Let $\sigma = (1\,3\,2)$ and $\tau = (2\,4\,3\,1)$ be elements of S_4. Find

(i) $(\sigma\tau)^{-1}$,

(ii) $\operatorname{sgn}(\sigma\tau^2)$,

(iii) $\rho \in S_4$ so that $\sigma\rho\tau = 1$,

(iv) the order of τ^2.