Review for Exam 3

1. Let p and q be positive prime numbers. List all positive divisors of pq^3.

2. Write -525 as a product of prime numbers.

3. Given k different prime numbers $p_1, \ldots, p_k$. Construct a prime number which is not on this list.

4. Write both $3 - 4i$ and $6 + 9i$ as products of Gaussian primes. Show also that the two given Gaussian integers are relatively prime.

5. Give all associates of 3 in

(a) in $\mathbb{Z}$,

(b) in $\mathbb{Z}[i]$,

(c) in $\mathbb{Z}[\sqrt{2}]$.

6. Factorize 61 into primes in the ring

(a) $\mathbb{Z}$,

(b) $\mathbb{Z}[i]$,

(c) $\mathbb{Z}[\sqrt{2}]$.

7. Factorize $28 + 16i \in \mathbb{Z}[i]$ into Gaussian primes.

8. Let $p \in \mathbb{Z}$ be a prime number and α, $\beta \in \mathbb{Z}[i]$ be Gaussian integers with β not being a unit. **Prove:** If $\alpha\beta = p^2$, then $N(\alpha) = p^k$ for some integer $k \leq 3$.

9. For all of the following $p \in \mathbb{Z}$ state whether p is a Gaussian prime or not. If not, give a factorization of p into two non-units.

(a) $p = 31$ **(b)** $p = 33$ **(c)** $p = 2$ **(d)** $p = 101$ **(e)** $p = 103$.

10. Consider the symmetric group S_7.

(a) Give all elements of S_7, which are in D_7, and represent them as product of disjoint cycles. Give also the sign $\operatorname{sgn} \sigma$ of all these elements.

(b) Can you give a way to write S_3 as a subgroup of S_7?

(c) Give an element of order 7 in S_7.

(d) Give a subgroup of order 4.

11. A k-cycle is a cycle of length k, i. e. a permutation of the form $(a_1\ a_2\ \dots a_k)$ with different a_i.
Prove: A k-cycle is its own inverse exactly when $k = 2$.

12. Let $\sigma = (2\,5\,3)$ and $\tau = (1\,2\,4)$ in S_5. Calculate

(a) $\tau\sigma$,

(b) $\sigma\tau\sigma^{-1}$,

(c) σ^{17},

(d) find $\alpha \in S_5$ so that $\alpha\sigma(1\,4) = (1\,2\,3)$.

13. Prove: If $G \subseteq S_n$ is a subgroup containing all permutations then $G = S_n$. $2) and (1\,2\,3\,4\,5)$,

14. In $\mathbb{Z}[\sqrt{2}]$ the following equation is true

$$(5 + 2\sqrt{2})(-3 + 5\sqrt{2}) = 5 + 19\sqrt{2} = (7 - 4\sqrt{2})(11 + 9\sqrt{2}).$$

(a) Calculate the norms of all factors given in the products on the left and on the right and show that these factors are prime elements in $\mathbb{Z}[\sqrt{2}]$.

(b) How does the result of (a) relate to the fact that $\mathbb{Z}[\sqrt{2}]$ is a unique factorization domain? Give the exact relationship between the two factorizations.

15. Consider the ring $\mathbb{Z}[2i] = \{a + 2bi \mid a,\, b \in \mathbb{Z}\}$, which can be regarded as a subset of $\mathbb{Z}[i]$.

(a) Determine the units of $\mathbb{Z}[2i]$,

(b) Show that $\mathbb{Z}[2i]$ is not a unique factorization domain by giving two appropriate factorizations of $4 \in \mathbb{Z}[2i]$.

16. Prove by induction: 5 divides $3^{4n} - 1$ for all $n \geq 1$.