Review for the Final Exam in Math 222.02

1. **Prove** that $\sqrt{7}$ is irrational.

2. Make sure that you know how to calculate in $\mathbb{Z}$, $\mathbb{Z}[i]$, $\mathbb{Z}[\sqrt{2}]$.
For example: Given $-1+5\sqrt{2},\ 16+11\sqrt{2} \in \mathbb{Z}[\sqrt{2}]$.

(a) Calculate a greatest common divisor.

(b) List all greatest common divisors.

(c) Find a Bézout-equation.

3. Show all associates of $2+3i \in \mathbb{Z}[i]$ in the Gaussian plane.

4. Find all roots of the polynomial $f = X^2 + X$ in $\mathbb{Z}_{10}$.

5. Let σ, $\tau \in S_n$ for some $n \geq 2$. What is the inverse of the element $\sigma^2\tau^3\sigma^{-1}$?

6. **Prove:** If b is a unit of a ring R, then b is not a zero divisor.

7. Given the polynomials $f = X^4 + 5X^2 + 3X + 1$ and $g = X^2 + 2X + 3$ with coefficients in $\mathbb{Z}_9$. Find polynomials a and r so that $f = ag + r$ and $\deg r < \deg g$.

8. Can you find a number $a + b\sqrt{2} \in \mathbb{Z}[\sqrt{2}]$ so that $(a+b\sqrt{2})(2-\sqrt{2}) = 1$?

9. Find $\frac{2}{5} \in \mathbb{Z}_{13}$ and $\frac{4}{9} \in \mathbb{Z}_{10}$.

10. Give an example of a ring R and two units a, $b \in R$, so that $a+b$ is not a unit.

11. **(a)** Determine the order of 2 in $\mathbb{Z}_{17}$.

(b) Why does $\mathbb{Z}_{17}\backslash\{0\}$ together with multiplication (modulo 17) form a group?

(c) Give a subgroup of order 4.

(d) Is 2 a square in $\mathbb{Z}_{17}$?

(e) Is -1 a square in $\mathbb{Z}_{17}$?

12. Factorize into Gaussian primes:

(a) $35i$,

(b) -31,

(c) $50 + 75i$,

(d) $3 + 15i$.

13. Consider the symmetric group S_6 and its subgroup D_6 of all symmetries of the regular hexagon.

(a) Find a subgroup of order 3 of D_6 and describe its action on the hexagon.

(b) List all flips of D_6 which have positive sign.

(c) Find the smallest subgroup of D_6 containing the rotation $(1\,2\,3\,4\,5\,6)$ and $(1\,3)(4\,6)$.

14. Consider the group $\mathbb{Z}_{18}$ with the addition $+$.

(a) Find all generators.

(b) Determine the order of the element 12.

(c) Give a subgroup of order 6 and determine all its cosets.

15. Prove: If G is a group with prime order $p = |G|$, then the only subgroups of G are $\{e\}$ and G.