Chris Bendel and Peter Cholak Math 222 - Sample Exam 1 Wednesday, February 24

Be sure to carefully write up your answers. Be sure to explain your answers.

(4 points each) **Define** the following terms:
a) $\sqrt[6]{4}$ (over the complex numbers).
b) The *argument* of a complex number z.
c) The *order* of an nth root of unity for a positive integer n.
d) The *additive inverse* of a number a in $\mathbb{Z}_n$ (n a postive integer).
(2 points each) Answer **True** or **False** - no work required:
a) $\arg(1+i) = \pi/4 + 2\pi n$ for any $n \in \mathbb{Z}$.
b) For any $z \in \mathbb{C}$, $|z|^3 = 3|z|$.
c) $3 \cdot 17 \equiv 3 \pmod 6$.
d) The coefficient of a^7b^4 in $(a+b)^{11}$ is 330.
(15 points) Find all solutions to $x^5 - 32 = 0$. Are these solutions constructible? Why or why not? (You do not need to simplify your answer to the form $a+bi$.)
(15 points) Find the multiplicative inverse of 14 in $\mathbb{Z}_{59}$.
(15 points) Prove that $n^{13} - n$ is divisible by 78 for any positive integer n.
(15 points) Exactly one of the following three problems will appear.

a) Let $\operatorname{lcm}(a,b)$ denote the *least common multiple* of a pair of integers a, b. Let ζ be the first nth root of unity for some positive integer n (i.e. $\zeta \neq 1$). Show that $o(\zeta^k) = \dfrac{\operatorname{lcm}(k,n)}{k}$ for $0 < k < n$.

b) Let n be a positive integer and ζ be the first nth root of unity (i.e. $\zeta \neq 1$) and let $\alpha = \zeta^k$ be any other nth root of unity. Prove that there exists an integer m such that $\zeta = \alpha^m$ if and only if k has a multiplicative inverse in $\mathbb{Z}_n$.

c) Prove that if n is an odd positive integer and ζ is any primitive nth root of unity, then so is ζ^2. Is this also true for n even? Why or why not?

(15 points) Exactly one of the following three problems will appear.

a) Prove that the least common multiple of any two positive integers k and m is $\dfrac{km}{(k,m)}$.

b) Let $p, p_1, p_2, p_3 \in \mathbb{Z}$ be prime numbers. Use Euclid's Lemma (Lemma 4.4) to prove that if p divides $n = p_1 \cdot p_2 \cdot p_3$, then p equals one of p_1, p_2, or p_3.

c) Let a and b be nonzero integers such that $g = (a, b)$. Prove that $\left(\dfrac{a}{g}, \dfrac{b}{g}\right) = 1$.