Peter Cholak and Juan Migliore Math 222 - Final Monday, May 7
(4 points each – 20 points total) **Define** the following terms:

A *primitive element* of the Galois field $GF(p, P(x))$.   The *index* of a

subgroup H of a finite group G.   A polynomial being *solvable by radicals*.

A *field*. (You can assume the definition of a ring.)   The *order* of an element

g in a group G.

(2 points each – 20 points total) Answer **True** or **False** - no work required: Every degree 3 polynomial over $\mathbb{Q}$ has a nonconstructible zero. There is a degree 5 polynomial over $\mathbb{Q}$ which is not solvable by radicals. The set of 2×2 matrices over the reals is an unital commutative ring. $\mathbb{Z}_6$ is a commutative ring but *not* an integral domain. There exists a field with 32 elements. If G is a group and H is a subgroup, then the identity element 1_G is an element of every coset of H. $(\mathbb{Z}_5, +)$ is a subgroup of $(\mathbb{Z}, +)$. The order of (1 2 3) is even in S_3. Every group of order 5 is cyclic.

Let H be a subgroup of the group G and let h be an element of H. Then the group $< h >$ is contained in H.

(10 points) Find all the zeros of the polynomial $x^5(x^3-4)-(x^3-4)$ in the complex numbers. Which of these solutions are not constructible? Why?

(15 points) Use the Euclidean algorithm to find $d = \gcd(304, 399)$ and then find integers x and y such that $d = 304x + 399y$.

(10 points) Given that $3i$ is a zero of $f(x) = x^3 - 6ix^2 - 11x + 6i$ over $\mathbb{C}$, find *all* the zeros of $f(x)$ in $\mathbb{C}$.

(15 points) Construct the cyclic table for $GF(3, x^2 + 2x + 2)$.

(10 points) Does S_8 have a cyclic subgroup of order 5? 11? 15? In each case, if so find such a subgroup. If not, explain why not.

(10 points) **(a).** Write down all the elements of A_4 *in disjoint cycle notation.*

(b). Find all the cosets of $\{id, (1\ 3)(2\ 4)\}$ in A_4.

(10 points) Find *all* subgroups of D_3.

(10 points) Let $F = GF(p, P(x))$ where $P(x)$ is irreducible over $\mathbb{Z}_p$. Show that for any a in F there is at most one pth root of a.

(10 points) Let R be a commutative ring with unity and let $U(R)$ denote the set of units of R. Show that $U(R)$ is a group under the multipliciation of R. (This group is called the group of units of R.)

(10 points) Show that every finite cyclic group is abelian (i.e. is commutative)