Math 222 Name: ________________________
Prof. Peter Cholak and Juan Migliore Friday, January 26

Solutions

Be sure to carefully write up your answers. It is suggested that you first write out a draft of your proposed questions and then carefully rewrite that draft to get your final version. You do *not* have to write the answers on this sheet of paper, but please staple all sheets together before turning your quiz in.

1. Show by induction that $n^4 - 4n^2$ is divisible by 3 for all $n \geq 1$.

First check it for $n = 1$: $1 - 4 = -3$ is divsible by 3. Now assume that $k^4 - 4k^2$ is divisible by 3 and consider

$$\begin{aligned} (k+1)^4 - 4(k+1)^2 &= [k^4 + 4k^3 + 6k^2 + 4k + 1] - 4[k^2 + 2k + 1] \\ &= [k^4 - 4k^2] + [4k^3 - 4k] + 6k^2 - 3 \\ &= [k^4 - 4k^2] + [4(k-1)(k)(k+1)] + [6k^2 - 3]. \end{aligned}$$

The first bracketed expression is divisible by 3 because of the inductive hypothesis. The second bracketed expression is divisible by 3 because it is 4 times the product of three consecutive integers. The last bracketed expression is obviously divisible by 3. Hence the whole expression is divisible by 3.

2. In the "real" world, we are used to the fact that a power of a non-zero number is again non-zero. We will later see that this fact is not true in all worlds. Show that the complex numbers do behave normally in this regard. Precisely, let z be an arbitrary complex number and suppose that we have $z^k = 0$ for some positive integer k. Show that z must be zero. (You may use the fact that you know it to be true for real numbers.)

If $z^k = 0$ then $0 = |z^k| = |z|^k$. But $|z|$ is a real number, and we know that a power of a non-zero real number is non-zero. Hence we have $|z| = 0$. But we've shown that this implies $z = 0$.

3. Find *all* the cube roots of $-8i$. Write them in the the form $a + bi$. Be sure to show your work.

Let $z = -8i$. Then $\arg(z) = \frac{3\pi}{2}$ and $|z| = 8$. We know that $\sqrt[3]{z}$ is a set with three elements. Each of the elements has modulus equal to $8^{\frac{1}{3}} = 2$. Let w be the first element in $\sqrt[3]{z}$. Then $\arg(w) = (\frac{3\pi}{2})/3 = \frac{\pi}{2}$. The other two roots then have arguments $\frac{\pi}{2} + \frac{2\pi}{3}$ and $\frac{\pi}{2} + \frac{4\pi}{3}$, i.e. $\frac{7\pi}{6}$ and $\frac{11\pi}{6}$. So

$$\begin{aligned} \sqrt[3]{-8i} &= \left\{2(\cos(\tfrac{\pi}{2}) + i\sin(\tfrac{\pi}{2})), 2(\cos(\tfrac{7\pi}{6}) + i\sin(\tfrac{7\pi}{6})), 2(\cos(\tfrac{11\pi}{6}) + i\sin(\tfrac{11\pi}{6}))\right\} \\ &= \left\{2i, -\sqrt{3} - i, \sqrt{3} - i\right\}. \end{aligned}$$