Peter Cholak and Juan Migliore Math 222 Wednesday, February 7

Be sure to carefully write up your answers. It is suggested that you first write out a draft of your proposed questions and then carefully rewrite that draft to get your final version. You do *not* have to write the answers on this sheet of paper.

Section 2.3: Problem 18: Explain why the solutions of the simultaneous equations $x^8+y^8=a$ and $x^8y^8=b$ have degree 2 algebraic expressions in $\{a,b\}$. *Hint:* Use Proposition 1.1.

Solution. Let $w=x^8$ and $z=y^8$. Then w and z are the solutions of the quadratic equation $X^2-aX+b=0$ (see Proposition 1.1). So both w and z have degree 2 algebraic expressions in $\{a,b\}$. On the other hand, $x\in\sqrt{\sqrt{\sqrt{w}}}$. So x has a degree 2 algebraic expression in $\{w\}$. Also, y has a degree 2 algebraic expression in $\{z\}$. This shows that the solutions of the above simultaneous equations have degree 2 algebraic expressions in $\{a,b\}$.

Show all the solutions of $x^6+9x^3+8=0$ are constructible. *Hint:* Solve it.

Solution. By the factorization, the equation becomes

$$(x^3+1)(x^3+8)=0$$

or further becomes

$$(x+1)(x^2-x+1)(x+2)(x^2-2x+4)=0$$

The solutions of this equation (hence of the original equation) are $-1,-2,\frac{1\pm\sqrt{3}i}{2},1\pm\sqrt{3}i$. All of them have degree 2 algebraic expressions in the integers and so they are constructible.

Show that a regular 30-gon is constructible. *Hint:* Make use of the angles in a regular 5-gon and a regular 6-gon, which are both constructible.

Solution. Note that a regular n-gon is constructible if and only if the complex number
$\cos\frac{2\pi}{n}+i\sin\frac{2\pi}{n}$ is constructible, and if and only if an angle of $\frac{2\pi}{n}$ is constructible. So we need only show the constructibility of the angle of $\frac{2\pi}{30}$. Since the angles in a regular 5-gon and a regular 6-gon are both constructible, the complex numbers $x=\cos\frac{2\pi}{5}+i\sin\frac{2\pi}{5}$ and $y=\cos\frac{2\pi}{6}+i\sin\frac{2\pi}{6}$ are both constructible. So

$$z=\frac{y}{x}=\cos(\frac{2\pi}{5}-\frac{2\pi}{6})+i\sin(\frac{2\pi}{5}-\frac{2\pi}{6})=\cos\frac{2\pi}{30}+i\sin\frac{2\pi}{30}$$

is also constructible. This implies that an angle of $\frac{2\pi}{30}$ is constructible, and hence a regular 30-gon is constructible.

Section 2.5: Problem 16. Prove that if n is a positive odd integer and ζ is a primitive nth root of unity, then so is ζ^2. Is this also true for even n? Justify your answer.

Solution. If ζ is a primitive nth root of unity then we have $o(\zeta)=n$, $\zeta^n=1$ and $\zeta^m\neq 1$ for $1\leq m<n$. Suppose ζ^2 were not a primitive nth root of unity, so $(\zeta^2)^m=\zeta^{2m}=1$ for some $m<n$. By Proposition 2.14, it follows that $2m$ is a multiple of $o(\zeta)=n$. But since $m<n$, we get $2m<2n$ so the only way a multiple of n could equal $2m$ is if $2m=n$. But this is impossible since n is odd.

This is not true for even n. For example, i is a primitive fourth root of unity but $i^2=-1$ is not.