Peter Cholak and Juan Migliore Math 222 Monday, March 19, 2001
Quiz 4

Be sure to carefully write up your answers. It is suggested that you first write out a draft of your proposed questions and then carefully rewrite that draft to get your final version. You do *not* have to write the answers on this sheet of paper.

Find the following examples. You do not have to prove your answers.

1. A ring that is not commutative.
2. A commutative ring that is not unital.
3. A commutative unital ring that is not an integral domain.
4. An integral domain that is not a field.
5. A field that contains the rational numbers $\mathbb{Q}$ and contains $\sqrt{3}$ but is not the whole set of real numbers.

Solution.

1. The matrix ring $M_2(\mathbb{R})$.
2. The set of even integers $2\mathbb{Z}$.
3. $\mathbb{Z}_6$.
4. The integers $\mathbb{Z}$.
5. The set of real numbers of the form $a+b\sqrt{3}$ where a and b are rational numbers. (Let's check this, just for fun. We first claim that this set does not contain $\sqrt{2}$, for example, so it is not the whole set of real numbers. Indeed, suppose that $\sqrt{2}=a+b\sqrt{3}$. Then
$$2 = a^2+3b^2+2ab\sqrt{3}, \quad i.e. \frac{2-a^2-3b^2}{2ab} = \sqrt{3}.$$
But the left-hand side is rational while the right-hand side is not. Contradiction.

   Clearly the sum or product of two elements in this set are again in this set. Since this set is contained in $\mathbb{R}$, it inherits all the properties of a field except inverses. But
$$\frac{1}{a+b\sqrt{3}} = \frac{a}{a^2-3b^2} - \frac{b}{a^2-3b^2}\sqrt{3}$$
so we are done.)

Let E be $\{a+b\sqrt{2}i \mid a,b \in \mathbb{Q}\}$, where i is the complex number with $i^2=-1$. We know that E is a ring with usual addition and multiplication.

(1) Show that E is a unital ring.

(2) For any nonzero element $x=a+b\sqrt{2}i \in E$, show that $x^{-1} \in E$.

(3) Show that E is a field.

Proof. (1) $1 = 1+0\cdot\sqrt{2}i \in E$ and $1\cdot x = x\cdot 1 = x$ for any $x \in E$. So 1 is

Let $p(x) = 2x^2 + 3x + 4$ and $q(x) = 4x^5 + 4x^4 + 4$. Working in $\mathbb{Z}_5[x]$ find $d(x)$ and $r(x)$ such that $q(x) = d(x)p(x) + r(x)$ and the degree of $r(x)$ is less than the degree of $p(x)$.

Solution: $d(x) = 2x^3 + 4x^2 + 2$ and $r(x) = 4x + 1$, as seen by the following long division:

$$
\begin{array}{rrrrrrr}
 & & & 2x^3 & +\,4x^2 & & +\,2 \\
\cline{2-7}
2x^2 + 3x + 4 & \sqrt{4x^5} & +\,4x^4 & + & & & +\,4 \\
 & 4x^5 & +\,x^4 & +\,3x^3 & & & \\
\cline{2-7}
 & & 3x^4 & +\,2x^3 & & & +\,4 \\
 & & 3x^4 & +\,2x^3 & +\,x^2 & & \\
\cline{2-7}
 & & & & 4x^2 & & +\,4 \\
 & & & & 4x^2 & +\,x & +\,3 \\
\cline{2-7}
 & & & & & 4x & +\,1
\end{array}
$$