**Math 223 Exam 1; February 20, 2004**

**Instructions for questions 1–4.** Answer the questions in the spaces provided. They are worth *5 points each.*

**1.** According to the division algorithm for $\mathbf{Z}$, what is the quotient and remainder when 55 is divided by $-8$?

**2.** Consider the expansion

$$(x+y)^8 = ax^8 + bx^7y + cx^6y^2 + \cdots$$

given by the binomial theorem. Express $a, b, c$ as binomial coefficients and determine their explicit numerical values (You do NOT have to evaluate the full expansion, just the part above involving $a$, $b$ and $c$).

**3.** Consider the permutations of the set $\{1,2,3,4\}$ given in two-line notation by

$$g = \begin{pmatrix} 1\,2\,3\,4 \\ 2\,3\,4\,1 \end{pmatrix}, \qquad h = \begin{pmatrix} 1\,2\,3\,4 \\ 3\,2\,1\,4 \end{pmatrix}$$

(recall this means $g(1) = 2, g(2) = 3, \ldots$ etc). Calculate the composite function $g \circ h$ in two-line notation, and then express your answer in cycle notation.

**4.** If functions $f, g\colon \mathbf{R} \to \mathbf{R}$ are defined by $f(x) = x - 1$ and $g(x) = x^2 - 1$ for $x \in \mathbf{R}$, compute $(g \circ f)(x)$ for $x \in \mathbf{R}$.

**Instructions for Questions 5–8** Give careful statements of the indicated definitions, principles or theorems in the spaces provided. These questions are worth *5 points each.* Points will be deducted for unclear or imprecise (or of course incorrect) answers.

**5.** State the principle of mathematical induction.

**6.** State the well-ordering principle for $\mathbf{N}$.

**7.** Carefully explain what is meant by saying that a function $f\colon A \to B$ is a bijection. How is the inverse function $f^{-1}\colon B \to A$ defined in terms of $f$?

**8.** Define what is meant by an injection, and state the Schroeder-Bernstein theorem.

**Instruction for Questions 9–12** Questions 9–12 are True/False questions worth *5 points each*. Write your answer (T or F) in the space provided. No working is required.

**9.** If $f\colon A \to B$ and $g\colon B \to C$ are functions such that $g \circ f$ is surjective, then $f$ is surjective.

Answer:

**10.** There is a function $f\colon \mathbf{R} \to \mathbf{R}$ such that $f(x) = |x - 1|$ if $x < 4$ and $f(x) = |x| - 1$ if $x > 2$.

Answer:

**11.** If $A$, $B$, $C$ are sets, then $A \cup (B \cap C) = (A \cup B) \cap (A \cap C)$ (Hint: use a Venn diagram).

Answer:

**12.** If $A$ and $B$ are finite sets and there is a surjection $f\colon A \to B$, then $|A| \geq |B|$, where $|X|$ denotes the size of a finite set $X$.

Answer:

**Instruction for Questions 13–15** Provide careful proofs of the indicated statements in the spaces provided. Points will be deducted for unclear or imprecise explanation.

**13.** *(15 points)* Prove that $\sqrt{2}$ is irrational.

**14.** *(10 points)* Prove: if $f\colon A \to B$, $g\colon B \to C$ and $h\colon C \to D$ are functions, then $(h \circ g) \circ f = h \circ (g \circ f)$.

**15.** *(15 points)* Prove by induction that for all $n \in \mathbf{N}$, $1 + 3 + 5 + \cdots + (2n - 1) = n^2$.