Math 223 Final Exam; May 3, 2004

Instructions for questions 1–8. Answer the questions in the spaces provided. They are worth *5 points each.*

1. Consider the permutations of the set $\{1, 2, 3, 4\}$ given in two-line notation by

$$g = \begin{pmatrix} 1\,2\,3\,4 \\ 3\,4\,1\,2 \end{pmatrix}, \qquad h = \begin{pmatrix} 1\,2\,3\,4 \\ 3\,2\,1\,4 \end{pmatrix}$$

(recall this means $g(1) = 3, g(2) = 4, \ldots$ etc). Calculate the composite function $g \circ h$, expressing your answer in two-line notation.

2. Find all integer solutions x, y of the linear Diophantine equation $30x + 24y = 18$.

3. Use limit laws to determine the limit of the sequence $\langle \frac{2 \cdot 4^n + 2^n}{3^n + 3 \cdot 4^n} \rangle$ as $n \to \infty$.

4. Determine the least upper bound of the subset A of $\mathbf{R}$ defined by

$$A := \{\, x \in \mathbf{R} \mid -1 \le x \le 2\} \cup \{3 - \frac{200}{n} \mid n \in \mathbf{N}\}.$$

5. Find all the distinct solutions (modulo 10) of the equation $x^2 - x - 2 \equiv 0 \pmod{10}$.

6. The function $f\colon \mathbf{R} \to \mathbf{R}$ is defined by $f(x) = x^3 - 2$. Determine whether f is a bijection and if so, give an explicit formula for the inverse function f^{-1}.

7. Calculate the inverse $[25]^{-1}$ of the congruence class of 25 in $\mathbf{Z}_{1001}$ (the integers modulo 1001).

8. Determine the integer a with $0 \leq a \leq 22$ and $2^{100} \equiv a \pmod{23}$.

Instructions for Questions 9–14 Give careful statements of the indicated definitions, principles or theorems in the spaces provided. These questions are worth *5 points each*. Points will be deducted for unclear or imprecise (or of course incorrect) answers.

9. Let $\sim$ be an equivalence relation on a set X. Give the definition of the equivalence class $[x]$ of an element x of X. If $a, b, c \in X$ with $c \in [a] \cap [b]$, what can you conclude about $[a]$ and $[b]$?

10. Explain what is meant by saying that a sequence $\langle a_n \rangle$ of real numbers converges to a limit L.

11. Explain what is meant by a lower bound, and the greatest lower bound, of a subset A of $\mathbf{R}$. Carefully state the greatest lower bound property of $\mathbf{R}$.

12. Give a careful definition of the notion of a Cauchy sequence.

13. State the Archimedean property of **R**.

14. State the Bolzano-Weierstrass theorem.

Instruction for Questions 15–18 Questions 15–18 are True/False questions worth *5 points each*. Write your answer (T or F) in the space provided. No working is required.

15. If $f\colon A \to B$ and $g\colon B \to C$ are functions such that $g \circ f$ is injective, then f is injective.

Answer:

16. Any subsequence $\langle b_n \rangle$ of a Cauchy sequence $\langle a_n \rangle$ is convergent.

Answer:

17. If $N \geq 2$ is any integer and $a \in \mathbf{Z}$ is not divisible by N, then $a^{N-1} \equiv 1 \pmod{N}$.

Answer:

18. The congrunce $ax \equiv 1 \pmod{p}$ has a unique solution (modulo p) for x if p is a prime number and a is an integer not divisible by p.

Answer:

Instruction for Questions 19–24 Provide careful proofs of the indicated statements in the spaces provided. Points will be deducted for unclear or imprecise explanation. Each question is worth *10 points*.

19. Prove by induction on n that $1 + 2 + 3 + \cdots + n = \frac{1}{2}n(n+1)$ for $n \in \mathbf{N}$.

20. Prove that if the sequence $\langle a_n \rangle$ is convergent, it is a Cauchy sequence.

21. Prove that $\sqrt{3}$ is irrational.

22. Prove that any convergent sequence is bounded.

23. State and prove the rational zeros theorem, on the rational zeros of a polynomial with integer coefficients.

24. Prove that if a, L are real numbers such that $a \geq L - \epsilon$ for all real numbers $\epsilon > 0$, then $a \geq L$.