Homework Set 1: Solutions

1.14: $[a,b] \cup [c,d] = [a,d] - (b,c)$.

1.17: The domain is $\mathbf{R}$ and the range is $[0,\infty)$.

1.25: Let a,b,c be the ages of the three daughters. Then a,b,c are natural numbers–i.e. we don't usually refer to people being 23.4 years old and certainly not -12 years old. Moreover, by rearranging we can assume that $a \geq b \geq c$. We're told that $abc = 36$, and since the only natural numbers that divide 36 evenly are $1,2,3,4,6,9,12,18,36$ that already narrows things down quite a bit. We have the following possibilities:

(1) $a = 36$, $b = c = 1$;
(2) $a = 18$, $b = 2$, $c = 1$;
(3) $a = 12$, $b = 3$, $c = 1$;
(4) $a = 9$, $b = 4$, $c = 1$;
(5) $a = 9$, $b = c = 2$;
(6) $a = 6$, $b = 6$ and $c = 1$;
(7) $a = 6$, $b = 3$, and $c = 2$;
(8) $a = 4$, $b = c = 3$.

Now the mother also refers to her 'oldest' daughter, which suggests that $a > b$ (i.e. $a \neq b$). This rules out the sixth possibility.

Finally, we're told that the sum of the ages is the house number (without being told the house number itself). Since the census taker (who did see the house number) was able to deduce the ages only *after* knowing that one daughter was older than the others, we can conclude that

$$a + b + c = 6 + 6 + 1 = 13.$$

The only remaining possibility for which this is true is the fifth. So the ages of the daughters are 9, 2, and 2.

1.33: To show that $S = T$, we must show that $S \subset T$ and that $T \subset S$—i.e. that each element of S is an element of T and vice versa. It's fairly easy to see that each element of T is an element of S—one just takes the ordered pairs from T one at time and verifies that they satisfy the inequality $(2-x)(2+y) > 2(y-x)$. For instance,

$$(2-1)(2+1) = 3 > 0 = 2(1-1),$$

so $(1,1) \in S$. Checking the other pairs in T is similar. So $T \subset S$.

To show that $S \subset T$, we observe that

$$\begin{aligned} (x,y) \in S &\Rightarrow \\ (2-x)(2+y) > 2(y-x) &\Rightarrow \\ 4 - 2x + 2y + xy > 2y - 2x &\Rightarrow \\ 4 > xy. & \end{aligned}$$

Now, since $x, y \in \mathbf{N}$ (and since the authors seem to forget that zero is a natural number!), we have $x \geq 1$. This means also that $y < 4/x \leq 4$, so $y \leq 3$. Similarly, $y \geq 1$ and $x \leq 3$. In summary

$$S \subset \{1,2,3\} \times \{1,2,3\} = \{(1,1),(1,2),(1,3),(2,1),(2,2),(2,3),(3,1),(3,2),(3,3)\}.$$

However, not all the elements in the righthand set actually belong to S. For instance, checking $(3,3)$ shows that

$$(2-3)(2+3) = -5 < 0 = 2(3-3),$$

so definitely $(3,3) \notin S$. Similar checking show that $(3,2),(2,3),(2,2) \notin S$. After we throw out these ordered pairs, we arrive at $S \subset \{(1,1),(1,2),(1,3),(2,1),(3,1)\} = T$.

Having shown $S \subset T$ and $T \subset S$, we can now conclude that $S = T$. $\square$

1.36: Showing that $S \subset T$ can be done by taking each ordered pair (e.g. $(2,3)$) from S and verifying that it satisfies $0 \leq 3x + y - 4 \leq 8$ (e.g. $3 \cdot 2 + 3 - 4 = 5$ which is between 0 and 8.) There's no need to write down all the details!

Now $S \neq T$—to show this, it's enough to exhibit just one element of T that does not belong to S. There are many possibilities here. For example $(1,4)$ certainly belongs to T but not to S. $\square$

1.41: Not much is needed here. Even a picture will suffice. Sufficient answers in words are

e: An element x belongs to $A \cap (B \cap C)$ if and only if x belongs to all three sets A, B, and C. The same goes for $(A \cap B) \cap C$. Hence $A \cap (B \cap C) = (A \cap B) \cap C$.

f: An element X belongs to $A \cup (B \cup C)$ if and only if x belongs to at least one of the three sets A, B, or C. The same goes for $(A \cup B) \cup C$. Hence $A \cup (B \cup C) = (A \cup B) \cup C$.

1.42: No because of leap years. Some years $f(\text{February}) = 28$, but other years $f(\text{February}) = 29$. So f is not well-defined.

1.45: The only non-function in the list is

b: For instance $-1 < 0 < 2$, so $f(0)$ could be either $|0-1| = 1$ or $|0| - 1 = -1$, so f is not well-defined.

1.47:

a: Let $x = (a+1)(a+2b)$. Note that $2b$ is always even. Hence $(a+2b)$ is even if a is even and odd if a is odd. On the other hand $a+1$ is even if a is odd and odd if a is even. In any case, x is the product of an even number and an odd number, so it follows that x is even. This means that $x/2 = (a+1)(a+2b)/2$ is an integer. Since $(a+1)$ and $(a+2b)$ are non-negative whenever a and b are, it follows that $(a+1)(a+2b)/2$ is a natural number. $\square$

b: The image of f is *all* of $\mathbf{N}$. To see that this is so, let $n \in \mathbf{N}$ be any number. It's enough to find just one pair (a, b) such that $f(a, b) = n$. In fact, $(a, b) = (0, n)$ will do:

$$f(0, n) = (0 + 1)(0 + 2n)/2 = n.$$

$\square$