Homework 11
(due Tuesday, April 27)

From the book: 13.30 (*hint: show the sequence is decreasing and bounded below*), 14.8, 14.14 (*hint: it is useful to show that there exists $N \in \mathbf{N}$ such that $n \geq N$ implies $b_n \geq M/2$*), 14.24a.

...and one more: here is a (really good) algorithm for computing square roots of positive numbers. Let $a \in \mathbf{R}$ be a positive real number, and define a sequence $< x >$ inductively by setting

- $x_1 = a$;
- for all $n \geq 1$, set $x_{n+1} = \frac{1}{2}(x_n + a/x_n)$.

Complete each of the following steps to show that $\lim_{n\to\infty} x_n = \sqrt{a}$.

(a) Prove that $x_n \geq 0$ for all n (*Hint: induction*).

(b) Prove $x_n^2 > a$ for all $n \in \mathbf{N}$ (*Hint: induction again, look at the difference between the quantities*).

(c) Prove that $< x >$ is decreasing.

(d) Now we know that $< x >$ converges. Why? Call the limit L.

(e) Show that $\lim_{n\to\infty} x_{n+1}$ is also L. That is, if we set $y_n = x_{n+1}$, then show that $< y >$ converges to L.

(f) Take limits of both sides of the formula for x_{n+1} to show that $L^2 = a$.

Use this algorithm (and a calculator) to compute $\sqrt{2}$ accurately to five decimal places. For your answer, it's enough to list all the x_n you compute along the way.