Solutions to Homework 2

2.2: The statement is false if it fails for even one choice of a and b. So let's take $a = 0$, $b = 1$. Suppose that m and n are integers such that $a = m + n$ and $b = m - n$. Then by adding the two equations together, we see that $2m = a + b = 1$. This means that $m = 1/2$ which contradicts our assumption that m is an integer. So we conclude that there are no integers m, n such that $m + n = 0$, $m - n = 0$. $\square$

To get a true statement, it's enough to require that $a + b$ is even. Then one can check that the numbers $m = (a + b)/2$ and $n = (a - b)/2$ will satisfy the given equations. Since $a + b$ is even, m will be an integer. Moreover, $a - b = (a + b) - 2b =$ "even - even" will also be even, and therefore n will be an integer, too.

2.4:

a) There exists $x \in A$ such that $x \geq b$ for all $b \in B$.

b) For all $x \in A$ There exists $b \in B$ such that $b \leq x$.

d) There exists $b \in \mathbf{R}$ such that $f(x) \neq b$ for all $x \in \mathbf{R}$.

2.8: Examples are legion. Here is one.

A. John is drinking.

B. John is driving.

C. John is breaking the law.

2.16a: Let's tackle the uniqueness part first: if $g, h : \mathbf{R} \to \mathbf{R}$ are functions satisfying

- $f(x) = g(x) + h(x)$,
- $g(x) = g(-x)$, and
- $h(x) = -h(-x)$

for all $x \in \mathbf{R}$, then we have that

$$f(-x) = g(-x) + h(-x) = g(x) - h(x),$$

too. So if add this equation to the one for $f(x)$, we get

$$f(x) + f(-x) = 2g(x)$$

which implies that $g(x) = (f(x) + f(-x))/2$ for all $x \in \mathbf{R}$. Similarly, we can subtract the equation for $f(-x)$ from the one for $f(x)$ and get

$$f(x) - f(-x) = 2h(x)$$

which implies that $h(x) = (f(x) - f(-x))/2$ for all $x \in \mathbf{R}$. In particular, both g and h are uniquely determined by f (in other words, once we know what f is, we can use the formulas we just derived to see exactly what g and h have to be). This completes the proof of uniqueness.

Now we tackle the *existence* of g and h. In a sense we've already done the hard work for this, having obtained candidate formulas for g and h. Now it's just a matter of checking to see that our candidates satisfy all the conditions listed above. So take $g(x) = \frac{1}{2}(f(x) + f(-x))$ and $h(x) = \frac{1}{2}(f(x) - f(-x))$. Then

- $g(x) + h(x) = \frac{1}{2}(f(x) + f(-x) + f(x) - f(-x)) = \frac{1}{2} \cdot 2f(x) = f(x)$;
- $g(-x) = \frac{1}{2}(f(-x) + f(x)) = \frac{1}{2}(f(x) + f(-x)) = g(x)$;
- $h(-x) = \frac{1}{2}(f(-x) - f(x)) = -\frac{1}{2}(f(x) - f(-x)) = -h(x)$.

So all conditions are satisfied, and our candidates for g and h actually work. $\square$

2.21: There exists an integer $n > 0$ such that $x < 1/n$ for every real number $x > 0$. The original (un-negated) version of the sentence is true—just take $x = 1/2n$, for example. That's a positive real number less than $1/n$.

2.22: There exist real numbers $x < y$ such that $f(x) > f(y)$.

2.30:

a) The tokens showing vowels must be turned over to make sure only odd numbers appear underneath them, and the tokens showing even numbers must be turned over to make sure that there are no vowels underneath them.

b) All tokens must be turned over.

2.32: If C is telling the truth, then so is B, which leaves only one liar. So C must be lying. If A is telling the truth, then A is a liar, which is absurd. So A is lying. If B is lying, then all three are liars, but that would mean that A was telling the truth after all. So B must be telling the truth.

The only possible scenario then is that A and C are liars while B is honest, and if you read their statements again you see that this possibility is consistent with what they say.

2.40: The key in both parts is to suppose you really can cover (what's left of) the checkerboard with the pieces shown. Then count how many white and how black squares got covered and find out what goes wrong.

a) Suppose (in order to obtain a contradiction) that there's a way to cover the checkerboard exactly with n copies of the given rectangle. Since adjacent squares are always opposite colors, this means that the board must contain exactly n white rectangles and n black rectangles. In particular, the board must contain the same number of white squares as it does black squares. But this contradicts the fact that, after diagonally opposing corners are removed, there are two more black than white squares. Hence our initial assumption was false: there is no way to cover the checkerboard perfectly with the given rectangle. $\square$

b) Again, in order to arrive at a contradiction, suppose that we can cover what's left of the checkerboard perfectly with T-shaped pieces. Each such piece contains four squares, and the area covered contains a total of $64 - 4 = 60$ squares. Hence, there are exactly 15 of the pieces covering the board. On the other hand, each T will cover either one black and three white squares or one white and three black squares. Suppose we have used n pieces of the first type and, therefore, $15 - n$ of the second type. Then, because there are the same number of white and black squares on the board, we see that

$$\begin{aligned} 3 \cdot n + 1 \cdot (15 - n) &= \text{white squares covered} \\ &= \text{black squares covered} = 1 \cdot n + 3 \cdot (15 - n). \end{aligned}$$

Solving this equation for n shows that

$$n = 15/2,$$

which contradicts the fact that n must be an integer. So our initial assumption was wrong again: there is no way to cover the board exactly with T shaped pieces. $\square$

2.47:

a) True. If x is odd, then $x = 2k + 1$ for some $k \in \mathbf{Z}$. Then $x^2 - 1 = 4k^2 + 4k = 4k(k + 1)$. Moreover, either k or $k+1$ is even, so $k(k+1)$ is even, and we can write $k(k+1) = 2n$ for some $n \in \mathbf{Z}$. Hence

$$x^2 - 1 = 4k(k + 1) = 8n$$

is evenly divisible by 8.

b) True. If $x^2 - 1$ is divisible by 8, then $x^2 - 1$ is even, which implies that x^2 is odd. Therefore, x itself is also odd.

2.51a: We first show that $A \cup (B \cap C) \subset (A \cup B) \cap (A \cup C)$. Let $x \in A \cup (B \cap C)$. Then $x \in A$ or $x \in (B \cap C)$. In the first case, it follows that $x \in A \cup B$ and $x \in A \cup C$, too. Hence $x \in (A \cup B) \cap (A \cup C)$. In the second case, it follows that $x \in B$ and $x \in C$. This implies that $x \in A \cup B$ and $x \in A \cup C$, and $x \in (A \cup B) \cap (A \cup C)$ as before. We conclude that

$$A \cup (B \cap C) \subset (A \cup B) \cap (A \cup C).$$

Now we show the "reverse inclusion" (use this phrase to sound suave in front of mathematicians) $(A \cup B) \cap (A \cup C) \subset A \cup (B \cap C)$. Let $x \in (A \cup B) \cap (A \cup C)$. Then x belongs to both $A \cup B$ and $A \cup C$. This could happen in one of two ways. First, it could be that $x \in A$. Second, it could be that $x \in B$ and $x \in C$—in other words, $x \in B \cap C$. Either way, $x \in A \cup (B \cap C)$. So we conclude

$$(A \cup B) \cap (A \cup C) \subset A \cup (B \cap C),$$

as desired.

Putting the two paragraphs together gives us

$$A \cup (B \cap C) = (A \cup B) \cap (A \cup C).$$

$\square$