Solutions to Homework 3

3.11. Let $S(n)$ denote the number of subsets of an n-element set. The first case is $n = 0$—i.e. the empty set. The only subset of the empty set is the empty set itself, so when $n = 0$, we see that $S(0) = 1 = 2^0$.

Now we move on to the induction step: supposing we know that $S(n) = 2^n$ when $n = k$, let us show that the same is true when $n = k+1$. Let A be a set containing $k+1$ elements. Let $x \in A$ be any element and observe that $B := A - \{x\}$ is a k-element set. Now A has two types of subsets: those that don't contain x and those that do. A subset is of the first type if and only if it is a subset of B. Therefore by our induction hypothesis, there are exactly 2^k subsets of the first type. Moreover, a subset is of the second type if and only if it has the form $\{x\} \cup C$, where $C \subset A$ does not contain x. But that means C is a subset of the *first* type. So that means there are 2^k possibilities for C and therefore 2^k subsets $\{x\} \cup C$ of the second type. All told, we see that

$$S(k+1) = 2^k + 2^k = 2^{k+1}.$$

This finishes the induction step.

We conclude that $S(n) = 2^n$ for all $n \in \mathbf{N}$. □

3.15. Let

$$S(n) = \sum_{i=1}^{n} (-1)^i i^2 = -1 + 2^2 - 3^2 + 4^2 + \dots (-1)^n n^2.$$

When $n = 1$, we have

$$S(1) = -1 = (-1)^1 \frac{1(1+1)}{2},$$

so the assertion is true in this case.

Now we proceed to the induction step: Suppose that the assertion is true when $n = k$—i.e. that

$$S(k) = (-1)^k \frac{k(k+1)}{2}.$$

We must then prove that the assertion is true when $n = k+1$—i.e. we must show that

$$S(k+1) = (-1)^{k+1} \frac{(k+1)((k+1)+1)}{2} = (-1)^{k+1} \frac{k^2+3k+2}{2}.$$

To do this, we relate $S(k+1)$ to $S(k)$ and use our induction hypothesis as follows.

$$\begin{aligned} S(k+1) &= \sum_{i=1}^{k+1} (-1)^i i^2 \\ &= (-1)^{k+1}(k+1)^2 + \sum_{i=1}^{k} (-1)^i i^2 \end{aligned}$$

$$\begin{aligned}
&= -(-1)^k(k^2+2k+1) + (-1)^k \frac{k(k+1)}{2} \\
&= (-1)^k \frac{-2(k^2+2k+1)+(k^2+k)}{2} \\
&= (-1)^k \frac{-k^2-3k-2}{2} \\
&= (-1)(-1)^k \frac{k^2+3k+2}{2} \\
&= (-1)^{k+1} \frac{k^2+3k+2}{2},
\end{aligned}$$

which is exactly what we wanted. This finishes the inductive step and allows us to conclude that

$$S(n) = (-1)^n \frac{n(n+1)}{2}$$

for all $n \in \mathbf{N}$. □

3.16. Let

$$S(n) = \sum_{i=1}^{n} i^3 = 1^3 + 2^3 + \ldots + n^3.$$

Initial case: When $n = 1$, we have

$$S(1) = 1^3 = 1 = \left(\frac{1(1+1)}{2}\right)^2.$$

Inductions step: Now suppose that the assertion is true when $n = k$—i.e. that

$$S(k) = \left(\frac{k(k+1)}{2}\right)^2.$$

Then on the one hand

$$\begin{aligned}
S(k+1) &= \sum_{i=1}^{k+1} i^3 = S(k) + (k+1)^3 \\
&= \left(\frac{k(k+1)}{2}\right)^2 + (k+1)^3 \\
&= \frac{k^2(k+1)^2 + 4(k+1)^3}{4} \\
&= \frac{k^4 + 6k^3 + 13k^2 + 12k + 4}{4}.
\end{aligned}$$

And on the other hand

$$\left(\frac{(k+1)((k+1)+1)}{2}\right)^2 = \frac{(k^2+2k+1)(k^2+4k+4)}{4} = \frac{k^4+6k^3+13k^3+12k+4}{4}.$$

Thus

$$S(k+1) = \left(\frac{(k+1)((k+1)+1)}{2}\right)^2,$$

so the assertion holds for $n = k+1$.

We conclude that

$$S(n) = \left(\frac{n(n+1)}{2}\right)^2$$

for all $n \in \mathbf{N}$. □

3.26. Initial step (n=1): we have by definition that $a_1 = 1$, which is certainly equal to $1^3 - 1 + 1 = 1$. So the assertion is true when $n = 1$.

Induction step: we assume that the assertion is true when $n = k$–i.e. that $a_k = k^3 - k + 1$. Given this assumption, we have on the one hand that

$$a_{k+1} = a_k + 3k(k+1) = k^3 - k + 1 + 3k^2 + 3k = k^3 + 3k^2 + 2k + 1,$$

and on the other hand that

$$(k+1)^3 - (k+1) + 1 = k^3 + 3k^2 + 3k + 1 - k - 1 + 1 = k^3 + 3k^2 + 2k + 1.$$

So $a_{k+1} = (k+1)^3 - (k+1) + 1$, as desired (did you ever really doubt it?), and the induction step is complete.

We conclude that $a_n = n^3 - n + 1$ for all $n \in \mathbf{N}$. □

3.33. Let's agree to count things like $[1, 1]$ as a closed interval, even though it only contains a single point. It'll affect the answer a little, but it won't really change the proof. As for the answer, I claim that the interval $[1, n]$ contains exactly $n(n+1)/2$ intervals with integer endpoints.

Proof.(by induction)

Initial step (n=1): in this case our interval is $[1, 1]$ and contains exactly one subinterval with integer endpoints—namely $[1, 1]$. Since $1 = 1(1+1)/2$, the assertion is true when $n = 1$.

Induction step: suppose we know that the interval $[1, k]$ contains exactly $k(k+1)/2$ subintervals with integer endpoints. I must show that the interval $[1, k+1]$ contains exactly $(k+1)((k+1)+1)/2$ subintervals with integer endpoints.

To do this, I observe that subintervals $I \subset [1, k+1]$ with integer endpoints come in two types:

1. subintervals $I \subset [1, k]$.
2. subintervals $I = [n, k+1]$ having $k+1$ as an endpoint.

Now by the induction hypothesis there $k(k+1)/2$ subintervals of the first type. And since n can range from 1 to $k+1$, there are exactly $k+1$ distinct subintervals of the second type. All told we have

$$\frac{k(k+1)}{2} + k + 1 = \frac{k^2 + 3k + 2}{2}$$

subintervals of $[1, k+1]$ with integer endpoints. On the other hand,

$$\frac{(k+1)((k+1)+1)}{2} = \frac{k^2+3k+2}{2}$$

works out to be the same number. So the assertion is true for the case $n = k+1$, concluding the induction step.

It follows that the assertion is true for all n. $\square$

3.38. Suppose that the goal of the game is to be the first to bring the total to $4n$ where $n \geq 1$. Then player two will always win provide she plays correctly.

Proof.(by induction) **Initial step (n=1):** If the goal is to be the first to reach $4 \cdot 1 = 4$, and player one first move is to play $m \in \{1, 2, 3\}$, then player two immediately wins by playing $4 - n$. So the above assertion is true when $n = 1$.

Induction step: Now suppose that player two can always win the game should the goal be to be the first to reach $4k$. We must show that player two will be able to win if the goal is to be first to $4(k+1)$. In fact, she can employ the following never fail strategy: by the induction hypothesis she'll can play so that she's the first to bring the total to exactly $4k$. But when that happens and the hapless player one adds $m \in \{1, 2, 3\}$ to the total, player two triumphs yet again by tacking on $4 - m$ and bringing the total to

$$4k + m + 4 - m = 4k + 4 = 4(k+1).$$

The induction step is now complete.

We conclude that player two will always win this game if she plays it right. $\square$

3.50. Assertion: If $f : \mathbf{N} \to (0, \infty)$ satisfies $f(1) = c$ and $f(x-y) = f(x)/f(y)$ for all $x, y \in \mathbf{Z}$, then $f(n) = c^n$ for all $n \in \mathbf{N}$. (Note that this covers the case where $c = 1$, too.)

Proof.(by induction)
Initial step (n=1): $f(1) = c = c^1$ is given to us as as a hypothesis.
Induction step: Supposing that $f(k) = c^k$, we have that

$$c^k = f(k) = f((k+1)-1) = f(k+1)/f(1) = f(k+1)/c$$

by hypothesis. Hence

$$f(k+1) = c \cdot c^k = c^{k+1},$$

completing the induction step.

We conclude that $f(n) = c^n$ for all $n \in \mathbf{N}$. $\square$