Solutions to 223 Homework

Practice: 4.1, 4.8, 4.9, 4.22, 4.24
To turn in: 4.2, 4.6, 4.12, 4.21 (assume n is odd), 4.25, 4.31, 4.34ab, 4.36a,
Extra credit: 4.21 (for even n), 4.39 (prove your formula is correct)

4.2. We have

$$\begin{aligned} 333_{(12)} &= 3*12^2+3*12^1+3*12^0 = 471 \\ 3333_5 &= 3*5^3+3*5^2+3*5^1+3*5^0 = 468 \end{aligned}$$

so the first number is larger.

4.6. f is not an injection because, for instance, $f(\text{Sunday}) = f(\text{Monday}) = 6$.

4.12.

a) False. For example $f(x) = e^x$ is decreasing, but -1 is not in $f(\mathbf{R})$.

b) False. For example the constant function $f(x) = 0$ for all $x \in \mathbf{R}$ is non-decreasing but definitely not injective!

c) False. For example the function $f : \mathbf{R} \to \mathbf{R}$ given by

$$f(x) = \begin{cases} 1/x & \text{if } x \neq 0 \\ 0 & \text{if } x = 0 \end{cases}$$

is injective but not monotone.

d) True.

Proof. (of contrapositive statement) If f is bounded, then there is a number $M \in \mathbf{R}$ such that $|f(x)| \leq M$ for all $x \in \mathbf{R}$. In particular, $f(x) \neq M+1$ for any $x \in \mathbf{R}$. So f is not surjective. $\square$

e) False. $f(x) = e^x$ is again a counterexample.

4.21. Suppose that n is odd. Then for each subset $S \subset [n]$ with an even number of elements, set

$$f(S) = [n] - S.$$

Then $f(S)$ has an odd number of elements. So f is a function from A to B. Moreover, f is bijective, because $f \circ g = g \circ f = \text{id}$ where $g : B \to A$ is also given by

$$g(S) = [n] - S.$$

4.25.

a) Not a surjection because $f(a,b) = a+b \geq 1+1 = 2$, so $1 \notin f(\mathbf{N}\times\mathbf{N})$.

b) A surjection: given $n \in \mathbf{N}$, we have $f(n,1) = n \cdot 1 = n$.

c) A surjection: once again, for any $n \in \mathbf{N}$, we have $f(n,1) = n \cdot 1 \cdot 2/2 = n$.

d) Not a surjection because $f(a,b) \geq (1+1)\cdot 1 \cdot (1+1)/2 = 2$ for all $(a,b) \in \mathbf{N}\times\mathbf{N}$. So $1 \neq f(a,b)$ for any $(a,b) \in \mathbf{N}\times\mathbf{N}$.

e) Not a surjection because (I claim) $2 \notin f(\mathbf{N}\times\mathbf{N})$. To see this, note that $f(1,1) = 1$, but if either a or b exceeds 1, then $f(a,b) \geq 1\cdot 2\cdot(1+2)/2 = 3$. Either way, $f(a,b) \neq 2$ for any $(a,b) \in \mathbf{N}\times\mathbf{N}$.

4.31. Suppose in order to obtain a contradiction that f^{-1} is not increasing on B. Then there are numbers $b_1 < b_2$ in B such that $f^{-1}(b_1) \geq f^{-1}(b_2)$. Now since f is a bijection, so is f^{-1}. In particular, f^{-1} is injective. This means that $f^{-1}(b_1) \neq f^{-1}(b_2)$. Hence

$$f^{-1}(b_1) > f^{-1}(b_2).$$

But then because f is increasing, we have

$$b_1 = f(f^{-1}(b_1)) > f(f^{-1}(b_2)) = b_2,$$

which contradicts our assumption that $b_1 < b_2$. Therefore f^{-1} is increasing. □

4.34ab.

a) True

Proof. It suffices to prove the contrapositive statement: "if f is not injective, then h is not injective." So:

If f is not injective, that means there are points $a_1, a_2 \in A$ such that $a_1 \neq a_2$ but $f(a_1) = f(a_2)$. This further implies that

$$h(a_1) = g(f(a_1)) = g(f(a_2)) = h(a_2).$$

So h is not injective. □

b) False.

Counterexample: Let $f : [0,\infty) \to \mathbf{R}$ be given by $f(x) = x$. Let $g : \mathbf{R} \to \mathbf{R}$ be given by $g(x) = x^2$. In particular g is not injective because $g(-x) = g(x)$ for all $x \in \mathbf{R}$.

Then $h : [0,\infty) \to \mathbf{R}$ is given by the same formula $h(x) = x^2$ as g, but h is injective because it's domain contains only *positive* real numbers, and $x \mapsto x^2$ is monotone increasing for $x > 0$.

4.36a.

a) Suppose that $f \circ g : A \to A$ is the identity function. Then for any $a \in A$, we have $f(b) = a$, where $b = g(a)$. Hence the image of f is all of A, and we conclude that f is surjective. $\square$