Homework 9

Book problems to turn in: 8.10, 8.12 (don't worry about the interpretation part, although it is a worthwhile thing to think about)

...and some others to turn in: Recall that we defined the rational numbers to be the equivalence classes a/b of pairs $(a, b) \in \mathbf{Z} \times (\mathbf{Z} - \{0\})$ with the equivalence relation

$$(a, b) \sim (c, d)$$

if and only if $ad = bc$.

1. Show by example that $\sim$ fails to be an equivalence relation if we allow $(a, b) \in \mathbf{Z} \times \mathbf{Z}$ (i.e. we allow b to be 0).

2. Here is an apparently simpler way (than the one we defined in class) to add ordered pairs $(a, b), (c, d) \in \mathbf{Z} \times (\mathbf{Z} - \{0\})$:

$$(a, b) + (c, d) := (a + b, c + d).$$

   Give examples illustrating the following two ways in which this definition is inappropriate for adding rational numbers.

   (a) The result of $(a, b) + (c, d)$ might not be in $\mathbf{Z} \times (\mathbf{Z} - \{0\})$.

   (b) If $(a, b) \sim (a', b')$ and $(c, d) \sim (c', d')$, then $(a, b)+(c, d)$ is not necessarily equivalent to $(a', b') + (c', d')$.

3. Show that the distributive law works for rational numbers. That is, if $a/b, c/d, e/f$ are rational, then

$$\frac{a}{b} \cdot (\frac{c}{d} + \frac{e}{f}) = \frac{a}{b} \cdot \frac{c}{d} + \frac{a}{b} \cdot \frac{c}{d}.$$

   Restated using ordered pairs instead of fractions, this becomes

$$(a, b) \cdot ((c, d) + (e, c)) \sim (a, b) \cdot (c, d) + (a, b) \cdot (e, f).$$

   Solve the problem by working with ordered pairs to prove this last statement.

4. Here is another way to show that the set $\mathbf{Q}$ of all rational numbers is countable. By the Schröder-Bernstein Theorem, it's enough to find injective functions $f : \mathbf{N} \to \mathbf{Q}$ and $g : \mathbf{Q} \to \mathbf{N}$. So...

   (a) Let $g : \mathbf{N} \to \mathbf{Q}$ be given by $g(n) = n/1$. Then g is injective because

$$g(m) = g(n) \to m/1 = n/1 \to m \cdot 1 = n \cdot 1 \to m = n.$$

   (b) Define $f_1 : \mathbf{Q} \to \mathbf{Z} \times \mathbf{N}$ by $f_1(x) = (p, q)$ where $x = p/q$, $q > 0$ and $\gcd(p, q) = 1$. Then f_1 is injective because p and q are uniquely determined by x (we proved this in class).

(c) Find a bijection $f_2 : \mathbf{Z} \times \mathbf{N} \to \mathbf{N} \times \mathbf{N}$ (remember how we defined a bijection from $\mathbf{Z}$ to $\mathbf{N}$ awhile back). OK, you finally have to do something! You don't need to show that your f_2 is a bijection, as long as it actually is one.

(d) Define $f_3 : \mathbf{N} \times \mathbf{N} \to \mathbf{N}$ by $f_3(m, n) = 2^m \cdot 3^n$. Prove that f_3 is injective. (Hint: Look in the book at the discussion about prime factorizations.)

(e) The function $f = f_3 \circ f_2 \circ f_1 : \mathbf{Q} \to \mathbf{N}$ is a composition of injective functions and therefore injective. We now have injective functions $g : \mathbf{N} \to \mathbf{Q}$ and $f : \mathbf{Q} \to \mathbf{N}$, so we're done. $\square$

Note that the only places where you have to do something is in parts (c) and (d).