1. Do each of the following (6 points each).

(a) Define *prime number.*

$p \in \mathbf{N} - \{1\}$ is prime if the only natural numbers that divide p are 1 and p.

(b) State the *well-ordering property.*

Every non-empty set $S \subset \mathbf{N}$ has a smallest element.

(c) State the contrapositive of the following assertion: *If $x_1 \neq x_2$, then $f(x_1) \neq f(x_2)$.*

If $f(x_1) = f(x_2)$ then $x_1 = x_2$.

(d) Without using words of negation, state the opposite (i.e. negation) of the following assertion: *For all $x \in A$ there exists $b \in B$ such that $b > x$.*

There exists $x \in A$ such that for every $b \in B$, we have $x \geq b$.

(e) Define *increasing function.*

Suppose that $A \subset \mathbf{R}$. A function $f : A \to \mathbf{R}$ is increasing if for all $x_1, x_2 \in A$, $x_1 < x_2$ implies that $f(x_1) < f(x_2)$.

2. Four of the following seven assertions are false. Identify three of them and give counterexamples on this page and (if necessary) the next. If you give counterexamples to more than three statements, I will simply grade the first three. Note that you do not have to justify your counterexamples—only present them. (10 points each)

(a) Let $f : A \to B$ and $g : B \to C$ be functions. If f is injective and g is injective then $g \circ g$ is injective. **True.**

(b) Let $f : A \to B$ and $g : B \to C$ be functions. If $g \circ f$ is injective then g is injective. **False.**

For example, $f : [0, \infty) \to \mathbf{R}$ given by $f(x) = \sqrt{x}$ and $g : \mathbf{R} \to \mathbf{R}$ given by $g(x) = x^2$.

(c) The function $f : \mathbf{N} \times \mathbf{N} \to \mathbf{N}$ given by $f(a, b) = a + b$ is surjective. **False.**

There is no $(a, b) \in \mathbf{N} \times \mathbf{N}$ such that $f(a, b) = 1$.

(d) If A, B, C are sets then $(A \cup B) \cap (A \cup C) = A \cap (B \cup C)$ **False.**

For example $A = \{1\}$, $B = C = \emptyset$.

(e) Let $S = \{(a, b) \in \mathbf{N} \times \mathbf{N} : 2a + b \leq 5\}$. Then $S \subset [2] \times [2]$. **False.**

$(1, 3) \in S$ but $(1, 3) \notin [2] \times [2]$

(f) Let A and B be sets and $f : A \to B$, $g : B \to A$ be injective functions. Then the cardinality of A is the same as that of B. **True.**

(g) Let A and B be sets. Suppose that A is a subset of B, but B is not a subset of A. Then the cardinality of A is not the same as the cardinality of B. **False.**

For example, $A = \mathbf{N} - \{1\}$ and $B = \mathbf{N}$.

3. On the remaining pages of this exam, do three of the following four problems. If you turn in solutions to all four, I will simply grade the first three. (13 points each)

(a) Let $n \in \mathbf{N}$ be a natural number whose base 3 decimal expansion is $122_{(3)}$. Give the base 7 decimal expansion of n.

We have

$$122_{(3)} = 1 \cdot 3^2 + 2 \cdot 3^1 + 2 \cdot 3^0 = 17 \text{ in base } 10,$$

which is equal to

$$2 \cdot 7^1 = 3 \cdot 7^0 = 23_{(7)}.$$

(b) Prove that there are infinitely many prime numbers.

Proof. Given in class. $\square$

(c) Let A, B, C be sets. Prove that $A - B \subset A - (B \cap C)$

Proof. Suppose that $x \in A - B$. Then $x \in A$ and $x \notin B$. Hence $x \notin B \cap C$. Therefore, $x \in A - (B \cap C)$. This shows that $A - B \subset A - (B \cap C)$. $\square$

(d) Prove by induction that $\sum_{i=1}^{n} 2n - 1 = n^2$.

Proof.(by induction on n)

Initial step: (n=1) We have

$$\sum_{i=1}^{1} 2i - 1 = 2 \cdot 1 - 1 = 1 = 1^2,$$

so the assertion holds when $n = 1$.

Induction step: Suppose the assertion is true when $n = k$—i.e. that

$$\sum_{i=1}^{k} 2i - 1 = k^2.$$

Then

$$\sum_{i=1}^{k+1} 2i - 1 = \left(\sum_{i=1}^{k} 2i - 1 \right) + 2(k+1) - 1 = k^2 + 2k + 2 - 1 = k^2 + 2k + 1 = (k+1)^2.$$

Hence the assertion holds when $n = k + 1$. This completes the induction step and the proof.
$\square$

4. How many points should Professor Diller give you for problem 4? (1 point total)

As many as he's willing.