Solutions to Homework 10

Book problems to turn in: 13.2, 13.8, 13.11, 13.21, 13.22, 13.25, 13.29

13-2. This is a colloquial version of the Archimedean principle (Theorem 13.9). Imagine that a is very small and b is very large in the statement of that theorem.

13-8. False. For example $S = \{0, 1\}$ (i.e. the set containing only 0 and 1) has $0 = \inf S$ and $1 = \sup S$.

13-11. Let $A = \lim_{n\to\infty} a_n$ and $B = \lim_{n\to\infty} b_n$.

(a) This is true. If $A < B$, then I choose

$$\epsilon = \frac{B - A}{2},$$

(which is positive—this is the important thing). By definition of limit, there are numbers $N', N'' \in \mathbf{N}$ such that

$$n \geq N' \;\Rightarrow\; |a_n - A| < \epsilon$$
$$n \geq N'' \;\Rightarrow\; |b_n - B| < \epsilon$$

Therefore, if $N = \max\{N', N''\}$ and $n \geq N$, I have (in particular) that

$$a_n - A < \frac{B - A}{2} \text{ and } B - b_n < \frac{B - A}{2}.$$

Rearranging these two inequalities gives

$$a_n < A + \frac{B - A}{2} = \frac{A + B}{2} = B - \frac{B - A}{2} < b_n.$$

So in summary, $a_n < b_n$ whenever $n \geq N$. □

(b) This is false. Consider, for example $a_n = 1 + 1/n$ and $b_n = 1 - 1/n$. Then

$$\lim_{n\to\infty} a_n = 1 \leq 1 = \lim_{n\to\infty} b_n,$$

but $a_n > b_n$ for all $n \in \mathbf{N}$.

13-21. Proof. Assume that the *least upper bound property* is true. I will show then that the *greatest lower bound property* is also true.

Let $S \subset \mathbf{R}$ be a non-empty set that is bounded below. Let $M \in \mathbf{R}$ be any lower bound. Let

$$T = \{x \in \mathbf{R} : -x \in S\}.$$

Then I claim that T is bounded above by $-M$: if $x \in T$, then $-x \in S$, so $-x \geq M$. Multiplying through by -1 gives $x \leq -M$.

Since T is non-empty and bounded above, the least upper bound property gives a least upper bound m for T. In particular, $m \leq -M$. It follows that $-m \geq M$, and we can use the same argument as in the previous paragraph to conclude that $-m$ is a lower bound for T. Since M was arbitrary we see that $-m$ is a lower bound for T that is at least as large as any other lower bound—in other words $-m$ is the greatest lower bound for T. So the *greatest lower bound property* holds.

Exactly the same reasoning shows that the *greatest lower bound property* implies the *least upper bound property.* □

13-22.

(a) Rearranging the inequality gives

$$x(x-5) < 0,$$

which is true if and only if the two factors have opposite signs. This happens when $x > 0$ and $x < 5$—i.e. $S = (0, 5)$. Therefore, $\sup S = 5$ and $\inf S = 0$.

(b) Rearranging the inequality gives

$$x(x-1)^2 > 0.$$

The squared term is positive for all x except 1, Hence $S = (0,1) \cup (1, \infty)$, so $\inf S = 0$, but S has no upper bound and therefore no least upper bound.

(c) This time, I rearrange and find

$$x(x^2 - 4x + 1) < 0$$

which will be true when the two factors have opposite signs. The quadratic term has roots $2 \pm \sqrt{3}$ and is positive for $|x|$ large. Hence, the quadratic term is positive for $x \in (2-\sqrt{3}, 2+\sqrt{3})$ and negative for $x \in (-\infty, 2-\sqrt{3}) \cup (2+\sqrt{3}, \infty)$. Since $2-\sqrt{3} > 0$, I conclude that $S = (-\infty, 0) \cup (2-\sqrt{3}, 2+\sqrt{3})$. Therefore $\sup S = 2+\sqrt{3}$, but $\inf S$ does not exist.

□

13-25. Let $\epsilon > 0$ be given, and set $N > \frac{1}{\epsilon^2 + 2\epsilon}$. If $n \geq N$, I have

$$\begin{aligned} |\sqrt{1+n^{-1}} - 1| &\geq |\sqrt{1+N^{-1}} - 1| \\ &> |\sqrt{1+\epsilon^2+2\epsilon} - 1| \\ &= |\sqrt{(1+\epsilon)^2} - 1| = \epsilon. \end{aligned}$$

So to summarize: I have shown that $n \geq N$ implies $|\sqrt{1+n^{-1}} - 1| < \epsilon$. This proves that

$$\lim_{n\to\infty} \sqrt{1+n^{-1}} = 1.$$

□

13-29. Proof. To see that $\langle x \rangle$ is monotone, note that

$$x_{n+1} - x_n = \frac{n+2}{2n+3} - \frac{n+1}{2n+1} = \frac{-1}{(2n+1)(2n+3)} < 0$$

for all $n \in \mathbf{N}$. That is, $x_{n+1} < x_n$ for all $n \in \mathbf{N}$, so the sequence is monotone decreasing. On the other hand, it's clear that

$$x_n = \frac{1+n}{1+2n} > 0$$

for all $n \in \mathbf{N}$. So the sequence is bounded below. It follows that the sequence $\langle x \rangle$ converges.

Now I will show that the sequence converges to 1/2. Given $\epsilon > 0$, I choose $N > \frac{1}{4\epsilon} - \frac{1}{2}$. Then if $n \geq N$, I estimate

$$\left| \frac{n+1}{2n+1} - \frac{1}{2} \right| = \frac{1}{4n+2} \leq \frac{1}{4N+2} < \frac{1}{4(\frac{1}{4\epsilon} - \frac{1}{2}) + 2} = \epsilon.$$

So $\lim_{n \to \infty} x_n = 1/2$. □