Solutions to Homework 11

Solutions

13-30. Proof. I claim that the sequence $< x >$ is increasing. To prove this, I check the difference

$$\begin{aligned} x_{n+1} - x_n &= \left(\frac{1}{n+2} + \ldots + \frac{1}{2n} + \frac{1}{2n+1} + \frac{1}{2n+2}\right) - \left(\frac{1}{n+1} + \frac{1}{n+2} + \ldots + \frac{1}{2n}\right) \\ &= \frac{1}{2n+1} + \frac{1}{2n+2} - \frac{1}{n+1} \\ &= \frac{n+1}{(2n+1)(2n+2)(n+1)} < 0, \end{aligned}$$

because $n > 0$ for all $n \in \mathbf{N}$. So $x_{n+1} < x_n$ for all $n \in \mathbf{N}$, justifying my claim.

Also for every $n \in \mathbf{N}$, there are n terms in the sum defining x_n, the largest of which is $1/(n+1)$. Therefore, I have the upper bound

$$x_n = \frac{1}{n+1} + \ldots + \frac{1}{2n} \leq n \cdot \frac{1}{n+1} < 1$$

So $< x >$ is a bounded monotone sequence and must converge by Theorem 13.16. □

14.8:

(a) This is true.

Proof. I will prove the contrapositive: if $< x >$ has limit L, then $< x >$ is bounded. To do this, I take $\epsilon = 1$. Since $< x >$ converges to L, there is an $N \in \mathbf{N}$ such that $|x_n - L| < 1$ for all $n \geq N$. In particular, $|x_n| < |L| + 1$ for all $n \geq N$. So if I set

$$M := \max\{|x_1|, \ldots, |x_{N-1}|, |L| + 1\} < \infty,$$

it then follows that $|x_n| \leq C$ for every $n \in \mathbf{N}$. That is, $< x >$ is bounded. □

(b) This is false. For example, if $x_n = (-1)^n/n$, then $< x >$ is neither increasing or decreasing, but it converges to 0.

14.14:

Proof. Let $\epsilon > 0$ be given. Since $a_n \to L$, there exists $N_1 \in \mathbf{N}$ such that $n \geq N_1$ implies

$$|a_n - L| < \frac{|M|\epsilon}{4}$$

Because $b_n \to M$ there exists $N_2 \in \mathbf{N}$ such that $n \geq N_2$ implies

$$|b_n - M| < |M|/2,$$

which is the same as saying, $M/2 < b_n < 3M/2$. Finally, I can also choose $N_3 \in \mathbf{N}$ such that $n \geq N_3$ implies

$$|b_n - M| < \frac{\epsilon M^2}{4|L|}$$

Now I set $N = \max\{N_1, N_2, N_3\}$. Then if $n \geq N$, all of the above inequalities concerning a_n and b_n are true, and I can estimate

$$\begin{aligned}
\left|\frac{a_n}{b_n} - \frac{L}{M}\right| &= \frac{|a_n M - L b_n|}{|b_n M|} \\
&= \frac{|a_n M - LM + LM - L b_n|}{|b_n M|} \\
&\leq \frac{|a_n M - LM| + |LM - L b_n|}{|b_n M|} \\
&= \frac{|a_n - L|}{|b_n|} + \frac{|L|}{|M b_n|}|b_n - M| \\
&\leq \frac{|a_n - L|}{|M|/2} + \frac{|L|}{M^2/2}|b_n - M| \\
&< \frac{|M|\epsilon}{4}\frac{2}{|M|} + \frac{2|L|}{M^2}\frac{\epsilon M^2}{4|L|} = \epsilon.
\end{aligned}$$

So to summarize, when $n \geq N$, I have shown that $|a_n/b_n - L/M| < \epsilon$. It follows that $\lim a_n/b_n = L/M$. □

14.24a: Since $\lim_{n\to\infty} x_{n+1} = \lim_{n\to\infty} x_n$ (see part (e) of the non-book homework problem), I have

$$L = \lim_{n\to\infty} x_n = \lim_{n\to\infty} x_{n+1} = \lim_{n\to\infty} x_n^2 - 4x_n + 6.$$

Then I can apply Theorem 14.5 to the right side of this equation, obtaining

$$L = L^2 - 4L + 6,$$

or $L^2 - 5L + 6$. It follows that $L = 0$ or 3.

...and one more: here is a (really good) algorithm for computing square roots of positive numbers. Let $a > 1$ be a real number, and define a sequence $< x >$ inductively by setting

- $x_1 = a$;
- for all $n \geq 1$, set $x_{n+1} = \frac{1}{2}(x_n + a/x_n)$.

Complete each of the following steps to show that $\lim_{n\to\infty} x_n = \sqrt{a}$.

(a) Prove that $x_n \geq 0$ for all n (*Hint: induction*).

Proof. *Initial Step.* When $n = 1$, $x_n = a > 0$ by hypothesis.

Induction Step. Suppose that $x_k > 0$. Then

$$x_{k+1} = \frac{1}{2}\left(x_k + \frac{a}{x_k}\right) > 0$$

since all quanities on the right side are positive.

I conclude that $x_n > 0$ for all $n \in \mathbf{N}$. □

(b) Prove $x_n^2 > a$ for all $n \in \mathbf{N}$ (*Hint: induction again, look at the difference between the quantities*).

Proof. *Initial Step.* $x_1^2 = a^2 > a$ since $a > 1$.

Induction Step. Suppose that $x_k^2 > a$. Then

$$x_{k+1}^2 - a = \frac{1}{4}\left(x_k + \frac{a}{x_k}\right)^2 - a = \frac{x_k^4 - 2ax_k^2 + a^2}{4x_k^2} = \frac{(x_k^2 - a)^2}{4x_k^2} \geq 0,$$

since $x_k^2 > a$.

I conclude that $x_n^2 > a$ for all $n \in \mathbf{N}$. □

(c) Prove that $< x >$ is decreasing.

Proof.

$$x_{n+1} - x_n = \frac{1}{2}\left(x_n + \frac{a}{x_n}\right) - x_n = \frac{x_n^2 - a}{2x_n} > 0,$$

because $x_n > 0$ and $x_n^2 > a$. Therefore $x_{n+1} < x_n$ for all n, and the sequence is decreasing. □

(d) Now we know that $< x >$ converges. Why? Call the limit L.

Answer. I have shown that $< x >$ is decreasing and bounded below by 0, so by Theorem 13.16, $< x >$ converges.

(e) Show that $\lim_{n\to\infty} x_{n+1}$ is also L. That is, if we set $y_n = x_{n+1}$, then show that $< y >$ converges to L.

Proof. Let $\epsilon > 0$ be given. Since $\lim x_n = L$, there exists $N \in \mathbf{N}$ such that $n \geq N$ implies $|x_n - L| < \epsilon$. But if $n \geq N$, so is $n + 1$. Hence $|x_{n+1} - L| < \epsilon$, too. It follows that $\lim x_{n+1} = L$. □

(f) Take limits of both sides of the formula for x_{n+1} to show that $L^2 = a$.

Proof. Since x_n and x_{n+1} both converge to L, I can use Theorem 14.5 to obtain

$$\begin{aligned} L &= \lim x_{n+1} = \lim \frac{1}{2}\left(x_n + \frac{a}{x_n}\right) \\ &= \frac{1}{2}\left(\lim x_n + \lim \frac{a}{x_n}\right) = \frac{1}{2}\left(L + \frac{a}{\lim x_n}\right) \\ &= \frac{1}{2}\left(L + \frac{a}{L}\right) \end{aligned}$$

(note in the fourth equality that $L^2 \geq a$ because $x_n^2 > a$; in particular $L \neq 0$). Rearranging this equation, I see that

$$L^2 = a.$$

That is, $\lim x_n = L = \sqrt{a}$. □

Use this algorithm (and a calculator) to compute $\sqrt{2}$ accurately to five decimal places. For your answer, it's enough to list all the x_n you compute along the way.

Answer.

- $x_1 = 2$.
- $x_2 = \frac{1}{2}(2 + 2/2) = \frac{3}{2} = 1.5$.
- $x_3 = \frac{1}{2}(3/2 + 4/3) = \frac{17}{12} = 1.416666\ldots$.
- $x_4 = \frac{1}{2}(17/12 + 24/17) = \frac{577}{408} = 1.4142156\ldots$.
- $x_5 = \frac{1}{2}(577/408 + 816/577) = 1.414235\ldots$.

So anyhow, it took me only 3 steps to reach an approximation of 1.4142 for $\sqrt{2}$.