Homework 9

Solutions to graded problems

8-10. Suppose first that m and n are relatively prime. I will prove by contradiction that

$$\frac{an+bm}{mn}$$

is in lowest terms. Suppose in order to obtain that contradiction that $p > 1$ is a common factor of mn and of $an+bm$. I can assume without loss of generality that p is prime (because p is at least divisible by a prime number). Thus $p|mn$ implies that $p|m$ or $p|n$—say for the sake of argument that $p|m$. Then in particular, p does not divide n, because m and n are relatively prime.

Now turning to the numerator, I see that $p|bm$ because $p|m$. And since $an = 1(an + bm) + (-1)bm$ is an integer combination of m and n, I obtain that $p|an$, too. So the fact that p is prime implies that $p|a$ or $p|n$. But I already know from the previous paragraph that p does not divide n. Hence $p|a$.

In summary $p|m$ and $p|a$. But this contradicts the (given) fact that a/m is in lowest terms. Hence no such p exists, and I conclude that $(an + bm)/mn$ is in lowest terms.

To go in the other direction ("$(an + bm)/mn$ in lowest terms implies that m and n are relatively prime") I prove the contrapositive: I suppose that m and n are not relatively prime and try to prove that $(an + bm)/mn$ is not in lowest terms. This is not so hard. If $p > 1$ divides both n and m, then it also divides all three products mn, an and bm. Hence $p|(an + bm)$, too, and I see that p is a common factor of both numerator and denominator in $(an + bm)/mn$. That is, the fraction is not in lowest terms. □

8-12. If $a/b < c/d$, then $ad < bc$ because b and d are positive. To prove that $a/b < \frac{a+c}{b+d}$, I must show that $a(b + d) < b(a + c)$. So I check the difference between the two sides:

$$a(b + d) - b(a + c) = ab + ad - ba - bc = ad - bc > 0,$$

because $ad > bc$. Hence $a(b + d) > b(a + c)$ as desired.

The proof that $\frac{a+c}{b+d} < c/d$ is similar. □

Extra problem 1. Note that any pair $(a, b) \in \mathbf{Z} \times \mathbf{Z}$ is related to $(0, 0)$: $a \cdot 0 = b \cdot 0$. Hence transitivity fails. For instance, $(1, 2) \sim (0, 0)$ and $(0, 0) \sim (2, 3)$, but $(1, 2) \not\sim (2, 3)$.

Extra problem 2. There are lots of right answers here.

(a) For instance $(1, 2) + (-1, -2) = (0, 0) \notin \mathbf{Z} \times (\mathbf{Z} - \{0\})$.

(b) For instance $(1,2) \sim (2,4)$ and $(2,3) \sim (2,3)$ but

$$(1,2) + (2,3) = (3,5) \not\sim (4,7) = (2,4) + (2,3).$$

Extra problem 3. On the one hand

$$(a,b) \cdot ((c,d) + (e,f)) = (a,b) \cdot (cf + ed, df) = (acf + aed, bdf).$$

On the other hand

$$(a,b) \cdot (c,d) + (a,b) \cdot (e,f) = (ac, bd) + (ae, bf) = (acbf + bdae, b^2df).$$

And finally,

$$(acf + aed)(b^2df) = ab^2cdf^2 + ab^2d^2ef = bdf(acbf + bdae).$$

So

$$(a,b) \cdot ((c,d) + (e,f)) \sim (a,b) \cdot (c,d) + (a,b) \cdot (e,f).$$

$\square$

Extra problem 4.

(a) Nothing to do.

(b) Nothing to do.

(c) Let

$$f_2(m,n) = \begin{cases} (m, 2n) & \text{if} \quad n > 0 \\ (m, -2n+1) & \text{if} \quad n < 0. \end{cases}$$

(d) Suppose that $f_3(m,n) = f_3(m',n')$. Then

$$2^m 3^n = 2^{m'} 3^{n'}$$

Hence both sides are prime factorizations of the same number. Since primes factorizations are unique, it follows that $m = m'$ and $n = n'$.

(e) Nothing to do.