Homework 5

From the book: 4.44, 4.47

My problems:
1. Let a, b, c, d be real numbers such that $a < b$ and $c < d$. Show that the closed intervals $[a, b]$ and $[c, d] \subset \mathbf{R}$ have the same cardinality.

2. Let $a, b \in \mathbf{R}$ be positive numbers. Use the Schroeder-Bernstein Theorem to show that the open interval $(0, a)$ has the same cardinality as the closed interval $[0, b] \subset \mathbf{R}$.

3. Prove that if n and m are natural numbers and $f : [n] \to [m]$ is surjective, then $n \geq m$. Suggestion: use induction on n. In the inductive step, there'll be two cases to consider—given $f : [k+1] \to [n]$, you might have $f(k+1) = n$ or you might have $f(k+1) < n$.

Extra credit: 4.48 (6 points for the formula for the bijection, 6 for a correct proof that the function is bijective)
Remark: In problem 4.47 and in my problem 1, you just have to give correct formulas for the bijections and their inverses—you don't have to prove that they're correct. Similarly, in my problem 2, you'll need to define two injections. You just have to give formulas that work—you don't actually have to prove they work.

Solutions

4.44. First I show that h is surjective. Let $k \in [m+n]$ be given. If $k \leq m$, then $k = f(x)$ for some $x \in A$, because $f : A \to [m]$ is surjective. Therefore, $k = h(x)$.

If $m < k \leq n$, then $0 < k - m \leq n$. So $k - m = g(x)$ for some $x \in B$, because $g : B \to [n]$ is surjective. Therefore, $h(x) = g(x) + m = k - m + m = k$. Either way, $k = h(x)$ for some $x \in A \cup B$. So h is surjective.

Now I show that h is injective. Supposing that $h(x_1) = h(x_2) = k$ for some $x_1, x_2 \in A \cup B$, I must prove that $x_1 = x_2$. I consider three cases

(i) $x_1, x_2 \in A$. In this case $h(x_1) = f(x_1)$ and $h(x_2) = f(x_2)$. So $f(x_1) = f(x_2)$. Since f is injective, I conclude that $x_1 = x_2$.

(ii) $x_1, x_2 \in B$. In this case $h(x_1) = g(x_1) + m$ and $h(x_2) = g(x_2) + m$. So $g(x_1) + m = g(x_2) + m$, which implies that $g(x_1) = g(x_2)$. Since g is injective, I conclude that $x_1 = x_2$.

(iii) $x_1 \in A$, $x_2 \in B$ (the case $x_1 \in B$, $x_2 \in A$ is similar). In this case $h(x_1) = f(x_1) \leq m$ and $h(x_2) = g(x_2) + m \geq m + 1$. In particular $h(x_1) \neq h(x_2)$, which contradicts our initial supposition. That is, this case never occurs.

In all three cases, we see that $h(x_1) = h(x_2)$ implies that $x_1 = x_2$. Hence h is injective.

Since h is surjective *and* injective, I conclude that h is bijective. □

4.47. Let $E \subset \mathbf{N}$ be the set of all even numbers and $O \subset \mathbf{N}$ the set of all odd numbers. Then $f(x) = 2x$ defines a bijection $f : \mathbf{N} \to E$ (with inverse function $f^{-1}(y) = y/2$) and $g(x) = 2x - 1$ defines a bijection $g : \mathbf{N} \to O$ (with inverse function $g^{-1}(y) = (y+1)/2$. Therefore both sets E and O have the same cardinality—i.e. E and O are both countable sets.

My problem 1: Let a, b, c, d be real numbers such that $a < b$ and $c < d$. Show that the closed intervals $[a, b]$ and $[c, d] \subset \mathbf{R}$ have the same cardinality.

Solution: The sets $[a, b]$ and $[c, d]$ have the same cardinality if and only if there is a bijection $f : [a, b] \to [c, d]$. The easiest way to find such a bijection is to first draw its graph. Then it becomes clear that a natural candidate for the graph is a straight line that joins the point (a, c) to the point (b, d). The function that produces this graph has the formula $f(x) = \frac{b-a}{d-c}(x - a) + c$ (if this looks complicated, it's only because of the fact that we don't have specific values for a, b, c, d; as far as the variable x is concerned, f just has the form 'mx + h', where m and h depend on a, b, c, d. Defined by this formula, $f : [a, b] \to [c, d]$ is a bijection with inverse function

$$f^{-1}(y) = \frac{d-c}{b-a}(y - c) + a.$$

So $[a, b]$ and $[c, d]$ have the same cardinality. □

My problem 2.: Let $a, b \in \mathbf{R}$ be positive numbers. Use the Schroeder-Bernstein Theorem to show that the open interval $(0, a) \subset \mathbf{R}$ has the same cardinality as the closed interval $[0, b] \subset \mathbf{R}$.

Solution: In order to use Schroeder-Bernstein I need an injection $f : (0, a) \to [0, b]$ and an injection $g : [0, b] \to (0, a)$. And again, it's easiest to approach this problem by first imagining what the graphs of f and g might look like. However, this time I'll skip the discussion and cut straight to the formulas for f and g.

The linear function $f(x) = \frac{b}{a}x$ satisfies $f(0) = 0$ and $f(a) = b$. So f maps the interval $(0, a)$ bijectively onto the interval $(0, b)$. And since $(0, b) \subset [0, b]$, it follows that $f : (0, a) \to [0, b]$ is injective.

To get an injective function $g : [0, b] \to (0, a)$ going the other way, I'll define a linear function that sends $[0, b]$ onto some interval *inside* $(0, a)$—say $[a/3, 2a/3]$ for instance. So I need a linear function $g(y) = my + k$ that sends 0 to $a/3$ and b to $2a/3$. The function

$$g(y) = \frac{a}{3b}y + \frac{a}{3}$$

does exactly this.

In summary, I've shown that there is an injection $f : (0, a) \to [0, b]$ and an injection $g : [0, b] \to (0, a)$. By the Schroeder-Bernstein Theorem, I conclude that there exists a bijection $h : (0, a) \to [0, b]$. Hence $(0, a)$ and $[0, b]$ have the same cardinality. □

My problem 3. Prove that if n and m are natural numbers and $f : [n] \to [m]$ is surjective, then $n \geq m$.

Solution: I work by induction on n.
Initial step: (n=1). If $f : [1] \to [m]$ is surjective, then there exists $x \in [1]$ such that $f(x) = 1$. As there's only one element in $[1]$, it must be that $x = 1$. So $f(1) = 1$. The same reasoning shows that $f(1) = m$. But $f(1)$ can't be two different things. Hence $m = 1$.
Induction step: Assume the assertion is true when $n = k$. That is, if $f : [k] \to [m]$ is surjective, then $k \geq m$. To complete the induction step I suppose that $f : [k+1] \to [m]$ is surjective and try to show that $k + 1 \geq m$. I consider two cases.

(i) $f(k + 1) = m$. From this it follows that $f : [k] \to [m - 1]$ is surjective (why?). Hence $k \geq m - 1$. That is, $k + 1 \geq m$.

(ii) $f(k + 1) < m$. Then by surjectivity there exists $x \in [k + 1]$ such that $f(x) = m$. Evidently $x \neq k + 1$, so $x \in [k]$. Using x, I define a function $g : [k] \to [m]$ as follows

$$g(t) = \begin{cases} f(t) & \text{if} \quad t \neq x \\ f(k+1) & \text{if} \quad t = x. \end{cases}$$

It follows that the image of g is the same as that of f except that $f(x) = m$. In other words the image of g is exactly $[m-1]$, so $g : [k] \to [m-1]$ is surjective. Once again our induction hypothesis leads us to conclude that $k - 1 \geq m$, or equivalently, $k + 1 \geq m$.

This completes the inductive step and the proof. □