1. Let R be the region bounded by

$$y = x, \quad xy = 2, \quad xy^3 = 1$$

Rewrite the integral ${}_R x^2y^2\,dA$ as an iterated integral in the variables $u = xy$ and $v = x/y$ (do not evaluate).

Solution:

Solve for x and y in terms of u and v:

$$u = xy \quad v = \frac{x}{y}$$
$$uv = xy\frac{x}{y} = x^2 \quad \frac{u}{v} = \frac{xy}{x/y} = y^2$$
$$x = (uv)^{1/2} \quad y = (u/v)^{1/2}$$

Compute the Jacobian:

$$\frac{\partial(x,y)}{\partial(u,v)} = \det \frac{1}{2}(uv)^{-1/2}v\frac{1}{2}(uv)^{-1/2}u\frac{1}{2}(\frac{u}{v})^{-1/2}\frac{1}{v}\frac{1}{2}(\frac{u}{v})^{-1/2}(-\frac{u}{v^2}) = \frac{1}{4}(uv\frac{u}{v})^{-1/2}(-\frac{u}{v}) - \frac{1}{4}(uv\frac{u}{v})^{-1/2}\frac{u}{v} = \frac{1}{4}\frac{1}{u}(-\frac{u}{v}) -$$

Transform the equations of R into equations in u and v:

$$y = x \to (uv)^{1/2} = (u/v)^{1/2} \to uv = u/v \to v^2 = 1 \to v = 1 xy = 2 \to u = 2 xy^3 = 1 \to (xy)y^2 = 1 \to u(u/v) = 1 -$$

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The new region is defined by

$$1 \le v \le u^2, \quad 1 \le u \le 2$$

$${}_R x^2y^2\,dA = \int_1^2 \int_1^{u^2} (uv)(u/v)\left|-\frac{1}{2v}\right| \,dv\,du = \int_1^2 \int_1^{u^2} \frac{u^2}{2v}\,dv\,du$$

Note: In this problem it is easier to compute the Jacobian 'backwards':

$$\frac{\partial(u,v)}{\partial(x,y)} = \det yx\frac{1}{y} - \frac{x}{y^2} = -\frac{x}{y} - \frac{x}{y} = -2\frac{x}{y} = -2v\frac{\partial(x,y)}{\partial(u,v)} = \left[\frac{\partial(u,v)}{\partial(x,y)}\right]^{-1} = -\frac{1}{2v}$$