1. The force of gravity on a spacecraft located at (x, y, z) is $F(x, y, z) = 3125/(x^2 + y^2 + z^2)$. Suppose the spacecraft's position and velocity at time $t = 1$ are $= 9 \supset +12$ and $\overset{\smile}{=}10 \subset -12 \supset +90$, respectively. Find $\frac{dF}{dt}$ at time $t = 1$.

Solution: $\frac{dF}{dt} = \underset{\cdot}{F}/dx\frac{dx}{dt} + \underset{\cdot}{F}/dy\frac{dy}{dt} + \underset{\cdot}{F}/dz\frac{dz}{dt} = \frac{-2(3125)}{r^4}(x\frac{dx}{dt} + y\frac{dy}{dt} + z\frac{dz}{dt})$ where $r^4 = (x^2 + y^2 + z^2)^2$. At $t = 1$, $(x, y, z) = (0, 9, 12)$ and $\frac{dx}{dt} \subset +\frac{dy}{dt} \supset +\frac{dz}{dt} = 10 \subset -12 \supset +90$ (given). Plug in to get $\frac{dF}{dt} = -120$.

2. Let $f(x, y) = x^2y - y^2z$. Compute the derivative of f at the point $(1, 2, 0)$ in the direction $(4, 0, 3)$.

Solution: $\overset{\smile}{=}(4 \subset +3)/\sqrt{16 + 9} = (4/5) \subset +(3/5)$. $f = 2xy \subset +(x^2 - 2yz) \supset -y^2 = 4 \subset + \supset -4$ at $(1, 2, 0)$. $D_{\check{f}} = f \cdot \overset{\smile}{=}16/5 - 12/5 = 4/5$.

3. Let $f(x, y, z) = e^{-yz}\cos(xy)$. Compute (f) at the point $(\pi, 1, 0)$.

Solution: $f = -y\sin(xy)e^{-yz} \subset -(x\sin(xy)e^{-yz} + z\cos(xy)e^{-yz}) \supset -ye^{-yz}\cos(xy) = -\sin(\pi) \subset -\pi\sin(\pi) \supset$ $-\cos(\pi) =$ at $(\pi, 1, 0)$.

4. Determine the equation of the plane tangent to the ellipsoid $x^2 + 2y^2 + 3z^2 = 20$ at the point $(3, 2, 1)$.

Solution: $f = 2x \subset +4y \supset +6z = 6 \subset +8 \supset +6$ at $(3, 2, 1)$. The equation of the tangent plane is then $6(x - 3) + 8(y - 2) + 6(z - 1) = 0$, or $3x + 4y + 3z = 20$.

5. Find the critical points of the function $f(x, y) = 2x^3y - 6xy + 3y^2$.

Solution: (a): $f_x = 6x^2y - 6y = 0$, (b): $f_y = 2x^3 - 6x + 6y = 0$. If $y = 0$ then (a) is OK and (b) implies $x = 0$ or $x = \pm\sqrt{3}$, and we get three critical points, $(0, 0)$, $(\pm\sqrt{3}, 0)$. If $y \neq 0$ then (a) implies that $x = \pm 1$; for $x = 1$, (b) implies $y = 2/3$ and for $x = -1$, (b) implies $y = -2/3$ so we get two more critical points, $(1, 2/3)$, and $(-1, -2/3)$.

6. Choose the statement below that applies to the function $f(x, y) = x^3y - xy^2 + 2xy$.

Solution: The given points to consider are $(\sqrt{2}, 0)$, $(0, 0)$ and $(0, 2)$. $f_x = 3x^2y - y^2 + 2y = 0$, $f_y = x^3 - 2xy + 2x = 0$. $f_y(\sqrt{2}, 0) \neq 0$ so $(\sqrt{2}, 0)$ is not a critical point. $f_x(0, 0) = 0 = f_y(0, 0)$ and $f_x(0, 2) = 0 = f_y(0, 2)$, so we must examine the discriminant to see what type of critical points they are. $D = (6xy)(-2x) - (3x^2 - 2y + 2)^2$; $D(0, 0) = -(2)^2 < 0$, and $D(0, 2) = -(-2)^2 < 0$, so both are saddle points.

7. Find the maximum value of the function $f(x, y) = (1 - y^2)\log(1 + x^2)$ on the rectangle $-1 \leq x \leq 1$, $-1 \leq y \leq 1$.

Solution: Critical points inside: (a): $f_x = \frac{1 - y^2}{1 + x^2}(2x) = 0$, (b): $f_y = -2y\log(1 + x^2) = 0$. (a) implies $x = 0$, or $y = \pm 1$. When $x = 0$, (b) implies $y = c$ (arbitrary); when $y = \pm 1$ (b) implies $x = 0$. At these critical points, $f(0, c) = 0$. On the boundary: $f(x, \pm 1) = 0$, and $f(\pm 1, y) = (1 - y^2)\log(2)$. The latter function has a maximum of $\log(2) = 0.693$ at $y = 0$.

8. Find the extreme values of the function $f(x, y) = 1 + x^2y$ on the unit circle $x^2 + y^2 = 1$.

Solution: Using Lagrange multipliers, solve (a): $2xy = \lambda 2x$, (b): $x^2 = \lambda 2y$, and (c): $x^2 + y^2 = 1$. If $x = 0$, then (a) is OK, and (c) implies $y = \pm 1$ ((b) implies $\lambda = 0$), so we get critical points $(0, \pm 1)$ and $f(0, \pm 1) = 1$. If $x \neq 0$ then (a) implies $\lambda = y$, (b) implies $x = \pm\sqrt{2}y$, and (c) implies $x = \pm\sqrt{1/3}$, so we get the critical points $(\pm\sqrt{2/3}, \pm\sqrt{1/3})$ and $f(\pm\sqrt{2/3}, \pm\sqrt{1/3}) = 1 \pm \frac{2}{3\sqrt{3}} = 1.385,\ 0,615$.

You could also plug in the constraint equation into f and find the maximum of $g(y) = f(\pm\sqrt{1-y^2}, y) = 1 + (1-y^2)y$: $g'(y) = 0$ when $3y^2 = 1$, etc.

9. Let R be the region between the x-axis and $y = x$ for $0 \le x \le 1$. Compute ${}_R 6ye^{x^3}\, dA$.

Solution: $\int_0^1 \int_0^x 6ye^{x^3}\, dy\, dx = \int_0^1 3x^2 e^{x^3}\, dx = e^{x^3}|_0^1 = e - 1 = 1.728.$

10. Reverse the order of integration of the integral $\int_1^5 \int_{\sqrt{y-1}}^2 f(x,y)\, dx\, dy$.

Solution: The region (described horizontally) is: $\sqrt{y-1} \le x \le 2$, $1 \le y \le 5$. Rewriting this vertically gives $1 \le y \le 1 + x^2$, $0 \le x \le 2$ (draw the picture) and the integral is $\int_0^2 \int_1^{1+x^2} f(x,y)\, dy\, dx$.

11. Find the area of the region inside the cardiod $r = 1 + \sin(\theta)$ in the first quadrant.

Solution: $\int_0^{\pi/2} \int_0^{1+\sin(\theta)} r\, dr\, d\theta = \int_0^{\pi/2} \frac{1}{2}(1+\sin(\theta))^2\, d\theta = \int_0^{\pi/2} \frac{1}{2} + \sin(\theta) + \frac{1}{2}\sin(\theta)^2\, d\theta = \int_0^{\pi/2} \frac{1}{2} + \sin(\theta) + \frac{1}{4} - \frac{1}{4}\cos(2\theta)\, d\theta = \frac{3}{4}\theta - \cos(\theta) - \frac{1}{8}\sin(2\theta)|_0^{\pi/2} = 1 + \frac{3\pi}{8}$.

12. Compute the volume of the solid region under the graph of $f(x,y) = 4 - x^2 - y^2$ over the triangular region defined by $x + y \le 1$ in the first quadrant.

Solution: The region is $0 \le y \le 1 - x$, $0 \le x \le 1$ (draw the picture). So the volume is given by $\int_0^1 \int_0^{1-x} 4 - x^2 - y^2\, dy\, dx = \int_0^1 4(1-x) - x^2(1-x) - \frac{1}{3}(1-x)^3\, dx = 4x - 2x^2 - \frac{1}{3}x^3 + \frac{1}{4}x^4 + \frac{1}{12}(1-x)^4|_0^1 = 4 - 2 - \frac{1}{3} + \frac{1}{4} - \frac{1}{12} = \frac{22}{12} = \frac{11}{6}$.

13. Find the average value of the function $f(x,y) = y\sin(xy)$ over the region $0 \le x \le \sqrt{\pi}$, $0 \le y \le \sqrt{\pi}$.

Solution: The region is a square with area $(\sqrt{\pi})^2 = \pi$, so the average is $\bar{f} = \frac{1}{\pi}\int_0^{\sqrt{\pi}} \int_0^{\sqrt{\pi}} y\sin(xy)\, dx\, dy = \frac{1}{\pi}\int_0^{\sqrt{\pi}} -\cos(xy)|_0^{\sqrt{\pi}}\, dy = \frac{1}{\pi}\int_0^{\sqrt{\pi}} -\cos(\sqrt{\pi}y) + 1\, dy = \frac{1}{\pi}(y - \frac{1}{\sqrt{\pi}}\sin(\sqrt{\pi}y))|_0^{\sqrt{\pi}} = \frac{1}{\pi}\sqrt{\pi} = \frac{1}{\sqrt{\pi}}$.

14. Suppose z is a function of u and v and that $u = x^2 - y^2$ and $v = \log(x-y)$. If $z_u(3,0) = -3$ and $z_v(3,0) = 5$, compute z/dx when $x = 2$ and $y = 1$.

Solution: $z_x = z_u u_x + z_v v_x = 2xz_u + \frac{1}{x-y}z_v$. Note that when $(x,y) = (2,1)$, we get $(u,v) = (3,0)$. Plug in the given values for x, y, z_u and z_v to get $z_x = 4(-3) + 1(5) = -7$.

15. Choose the function below that has the following graph.

Solution: The graph clearly has more than one critical point (you should be able to see a local minimum and two saddle points). All of the given functions have only one critical point except $f(x,y) = y^2 + x^2y - y$ which has three critical points: $(0, 1/2)$ and $(\pm 1, 0)$.
You could also examine the edges: When $y \approx 1$, the graph looks like a parabola opening up and when $y \approx -1$, the graph looks like a parabola opening down. Only two of the given answers give something like $x^2 + c$ when $y = 1$ and only one of these (the above answer) looks like $-x^2 + c$ when $y = -1$.