amsppt

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Mathematics 226, Solutions to Final Exam
Differential Equations and Linear Algebra
Fall Semester
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1. a) Solve the initial value problem

$$y' - xy + 2x = 0, y(0) = 0.$$

Solution: We have $y' - xy = -2x$ or $(e^{x^2}2)' = -2xe^{-x^2}2$ or $e^{-x^2}2y = \int -2xe^{-x^2 2}dx + c = 2e^{-x^2}2 + c$ or $y = 2 + ce^{-x^2}2$. Since $0 = y(0) = 2 + c$, we have $c = -2$ and $y = 2 - 2e^{-x^2 2}$

b) Solve the DE

$$(xe^{xy} + 4y^3)dydx = 2x - ye^{xy}$$

Solution. We have

$$M \to ye^{xy} - 2x + N \to (xe^{xy} + 4y^3) = 0$$

Since $\delta M \delta y = e^{xy} + xye^{xy} = \delta N \delta x$ the DE is exact. Its solution is of the form $4(x, y) = C$ where $\psi(x, y) = \int M dx = \int (ye^{xy} - 2x)dx = e^{xy} - x^2 + h(y)$. We must have $\gamma\psi\delta y = N$ or $xe^{xy} + dhdy = xe^{xy} + 4y^3$. Thus, $dhdy = 4y^3$ or $h(y) = y^4$. Therefore the solution of the DE is given by $e^{xy} - x^2 + y^4 = C$.

2. Given that $y_1 = e^x$ and $y_2 = x^2e^x$ are two solutions to the DE

$$xy'' - (2x + 1)y' + (x + 1)y = 0,$$

find the general solution to the DE

$$xy'' - (2x + 1)y' + (x + 1)y = x(x^2 - 2x - 1)e^x.$$

Solution. A particular solution of the non-homogeneous equation is of the form $y_p = y_1u_1 + y_2u_2$ where u_1 and u_2 satisfy the system

$$y_1u_1' + y_2u_2' = 0 \qquad e^xu_1' + x^2e^xu_2' = 0$$

or

$$y_1'u_1' + y_2'u_2' = (x^2 - 2x - 1)e^x, \quad e^xu_1' + (2x + 1)e^xu_2' = (x^2 - 2x - 1)$$

or $u_1' + x^2u_2' = 0$

3. Let $y(x) = \sum_{n=0}^{\infty} a_n(x - 3)^n$ be the power series expansion of the solution to the initial value problem

$$y'' - 2(x - 3)y' - 2y = 0, y(3) = 0, y'(3) = 1.$$

a) Find the recurrence relation for the coefficients a_n. **Solution.** We must have $\displaystyle\sum_{n=2}^{\infty} n(n-1)a_n(x-3)^{n-2} - 2(x-3)\sum_{n=1}^{\infty} na_n(x-3)^{n-1} - 2\sum_{n=0}^{\infty} a_n(x-3)^n = 0$ **or** $\displaystyle\sum_{n=0}^{\infty}(n+2)(n+1)a_{n+2}(x-3)^n - 2\sum_{n=0}^{\infty} a_n(x-3)^n = 0$ **or** $\displaystyle\sum_{n=0}^{\infty}\left[(n+2)(n+1)a_{n+2} - 2na_n - 2a_n\right](x-3)^n = 0$ **or** $(n+2)(n+1)a_{n+2} - 2na_n - 2a_n = 0, \quad n = 0, 1, ...$ **or** $(n+2)(n+1)a_{n+2} - 2(n+1)a_n = 0, \quad n = 0, 1, 2....$ **or** $a_{n+2} = 2n + 2a_n, \quad n = 0, 1, 2, ..$

b) Then compute all a_n and write the solution $y(x)$. What is its radius of convergence? **Solution.** Since $a_0 = y(2) = 0$ and $a_1 = y'(2) = 1$, we have $a_2 = a_0 = 0, a_4 = 24a_2 = 0, ..., a_{2n} = 0, ...a_3 = 23a, = 23, a_5 = 25a_3 = 2325 = 2^2 3.5, a_7 = 27a_5 = 272^2 3 \cdot 5 = 2^3 3 \cdot 5 \cdot 7, ..., a_{2n+1} = 2^n 3 \cdot 5 \cdots (2n+1)$ Thus $y(x) = \sum_{n=0} \infty 2^n 3 \cdot 5 \cdots (2n+1)(x-3)^{2n+1}$
The radius of convergence of y is $R = \infty$.

4. Let

$$A = (\,1\,)\,3 - 52 - 7$$

a) Find the reduced echelon form of A. **Solution.** a) By row elementary operations A are reduced to

$$(\,1\,)\,3 - 52 - 7$$

$$(\,1\,)\,3 - 501$$

b) Find the solution space W of the system $Ax = 0$ and a basis for W. **Solution.** The system $Ax = 0$ is reduced to

$$x_1 \quad - 2x_3 \quad - 5x_5 = 0$$

Let $x_5 = \alpha$, $x_3 = \beta$ then $x_4 = 3\alpha$, $x_2 = \beta - 2\alpha$ and $x_1 = 5\alpha + 2\beta$ and

$$W = \{(\,x\,)_1$$

A basis for W is $\{(\,5\,)$

c) Rank of $A = 3$ dim $(W) = 2$

5. Use Cramer's Rule to solve the system:

$$3x_1 + x_2 \qquad = 1$$

Solution. The determinant Δ of the coefficient is $\Delta = 310$

$$\Delta_1 = 110$$

$$\Delta_2 = 310$$

6. Find the general solution of the system of the DE

$$dxdt = (\,1\,)\,2$$

Solution. Let $x = \xi e^{\lambda t}$ where $\xi = (\xi_1 \xi_2)$ is a constant vector and λ is a constant. For x to be a solution we must have

$$\lambda e^{\lambda t} \xi = e^{\lambda t} A \xi$$

The last system has non-zero solutions if $|A - \lambda I| = 0$ or $1 - \lambda 2$
If $\lambda = 5$ then ξ satisfies the system $(-)\, 42$
Therefore, the general solution for our system of DE is given by

$$(\, x \,)_1$$

7. Assume that the human population $p = p(t)$ of the earth obeys the logistic equation

$$dpdt = r(1 - pk)p, \;\; r = 0.029, \;\; k = 10^{10}.$$

If the population now is 1210^{10}, find the population $p(t)$ at any time t. Also find $\lim_{t \to \infty} p(t)$.

8. Find all terms up to degree four of the Taylor series expansion at $x_0 = 0$ of the function $y = y(x)$, where y is the solution to the following initial value problem

$$y'' - xy' - e^{2y} = 0, \;\; y(0) = 0, \;\; y'(0) = 1.$$