amsppt

1. Find the solution of the following initial value problem:

$$y' + \cos x\ y = \cos x, \quad y(0) = 2$$

Solution The integrating factor is $\mu(x) = e^{\int cosxdx} = e^{sinx}$ After multiplying by e^{sinx} the eqn becomes $e^{sinx}y)^1 = cosxe^{sinx}$ Thus $e^{sinx} = \int cosxe^{sinx}dx + c = \int e^{sinx}d(sinx) + c = e^{sinx} + c$ and $y(x) = 1 + ce^{-sinx}$ since $2 = y(0) = 1 + ce^{\circ} = 1 + c$ We have $c = 1$ and $y(x) = 1 + e^{-sinx}$

2. A certain population increases at a rate proportional to the square root of the population present. At the beginning the population was 100. One hour later it is 900.

F ind the population $p(t)$ at any time.

F ind the population four hours later;i.e. find $p(4)$.

Solution

$$dpdt = r\sqrt{p}$$

$$\Rightarrow p^{-12}dp = rdt \Rightarrow p^{-1/2+1}-1/2+1 = rt + c \Rightarrow \sqrt{p} = 12(rt + c)$$

$$10 = \sqrt{100} = \sqrt{p(0)} = 12(r(o) + c) =\Rightarrow c = 20$$

$$30 = \sqrt{900} = \sqrt{p(1)} = 12(r(1) + 20) \Rightarrow r = 40 \Rightarrow P(t) = 100(2t + 1)^2$$

$$P(4) = 100(2(4) + 1)^2 = 100(81)$$

$$P(t) = 100(2t + 1)^2$$

$$P(4) = 8,100$$

3. A body with weigt 160 pounds falls from rest in a medium offering resistance equal to $10v^2$. Find the speed $v(t)$ of the body at any time t. Also find $\lim_{t\to\infty} v(t)$.

Solution $mdudt = mg - 10u^2$, $m = wg = 16032 = 5$ $dudt = 32 - 2u^2$ $du16 - v^2 = 2dt \Rightarrow$ $\int du16 - v^2 = 2t + c$ $18 \int [14 - u + 14 + u]du = 2t + c$ $18en(4 + u) - 18en(4 - u) = 2t + c$

$en4 + u4 - u = 16t + c$ At $t = 0$: $ln4 + 04 - 0 = c$ or $ln1 = c = 0$ Thus $4 + u4 - u = e^{16t}$
$u(t) = 41 - e^{-16t}1 - e^{-16t}$

4. Find the general solution of the D.E.

$$y'' - 6y' + 9y = e^{3x} + \sin x.$$

Solution First we solve the homogeneous equation

$$y'' - 6y' + 9y = 0.$$

To solve this we need to solve the quadratic $r^2 - 6r + 9 = 0$. We find $r = 3$ and the solutions to the homogeneous equation are

$$y_c = c_1 e^{3x} + c_2 x e^{3x}$$

A particular solution of the non-honogeneous equation is given by

$$y_p = Ax^2 e^{3x} + BsinX + CcosX$$

From the D.E. we obtain

$$A[2e^{3x} + xe^{3x} + 9x^2 e^{3x}] - 6A[2xe^{3x} + 3x^2 e^{3x}] + 9Ax^2 e^{3x} + [-BsinX - CcosX] - 6[BcoxX - CsinX] + 9[BsinX$$

We must have $2A = 1,\ 8B + 6C = 1, and\ 8C - 6B = 0$

or $A = 12,\ B = 225,\ C = 350$. Therefore,

$$y = y_c + y_p = e^{3x}(c_1 + c_2 x + 12x^2) + 225sinX + 350CosX$$

5. Solve the differential equation

$$y'' - 2y' + y = 2e^x \ln x, \quad x > 0.$$

Solution The solution of the D.E. is given by $y = y_c + y_p$ where y_c is the solution to $y'' - 2y' + y = 0$ and y_p is a particular solution. We have $r^2 - 2r + 1 = (r-1)^2 = 0$ or $r = 1$. Thus,

$$y_c = c_1e^x + c_2xe^x.$$

y_p is given by $y_p = e^xu_1 + xe^xu_2$ where

$$e^xu_1' + xe^xu_2' = 0$$

$$e^xu_1' + (x+1)e^xu_2' = 2e^xlnx$$

we have

$$u_1' = 0xe^x$$

e$^x xe^x$

= -2xe$^{2x}ln\ xe^2x = -2x\ ln\ x$

or,

$$u_1 = -\int 2x\ ln\ x\ dx = -\int ln\ x\ d(x^2) = -x^2\ ln\ x + \int x^21xdx$$

or

$$u_1 = -x^2\ ln\ x + x^22$$

$$u_2' = e^x0$$

e$^x xe^x$

= 2e$^{2x}lnxe^{2x} = 2\,ln\ x$

or

$$u_2 = 2 \in ln\,x\,dx = 2x\,ln\,x - 2 \in 1xx\,dx = 2x\,ln\,x - 2x.$$

Therefore,

$$y_p = e^x(-x^2\,ln\,x + x^22) + xe^ax(2x\,ln\,x - 2x)$$

and

$$y = e^x[c_1 + c_2x - x^2 ln\, x + x^2 2 + 2x^2\, ln\, x - 2x^2]$$

or

$$y = e^x[c_1 + c_2x + x^2\, ln\, x - 32x^2]$$

6. Find the general solution to the D.E.

$$y'' - 4y' + 4y = 2e^{2x}$$

Solution The general solution of the D.E. is given by $y = y_c + y_p$ where y_c is the general solution of the homogeneous equation and y_p is a particular solution. Since $r^2 - 4r - 5 = 0$ implies $r = 2$ is a double root, we have

$$y_c = ce^{2x} + c_2xe^2x$$

y_p is of the form $y_p = Ax^2e^{2x}$. From the D.E. we obtain

$$(2e^{2x} + 8xe^{2x} + 4x^2e^{2x}) - 4(2xe^{2x} + 2x^2e^{2x)+4x^2e^{2x}=2e^{2x}}$$

Therefore, $A = 1$ and

$$y = (x^2 + c_2x + c_1)e^{2x}$$

b) Show that any solution to the D.E. is positive for all **x** with $|x|$ large enough.

Solution By part (a), a solution to the D.E. is given by

$$y(x) = (x^2 + bx + c)e^{2x},\ b, c \in R.$$

Since $e^{2x} > 0$ for all x and since for $x \neq 0$ we have

$$x^2 + bx + c = x^2\left(1 + bx + cx^2\right) \geq x^2\left(1 - |b||x| - |c|x^2\right) \geq x^2\left(1 - M|x| - M^2x^2\right)$$

where $M = max\{|b|, |c|, 1\}$.

Thus, if $M|x| < 12$ or $2M < |x|$, then $x^2 + bx + c > x^2\,(1 - 12 - 14) = 14x^2 > 0$.

Thus, if $|x| > 2M$ then $y(x) > 0$.

7. Example $(Variation of Parameters) 1991 Find the general solution to the D.E. y'' + y = 1cos^3 x\,,\ -n2 < x < n2$

Solution $The general solution to D.E. is given by \mathrm{y} = \mathrm{y}_c + y_p$ where y_c is the general solution to the homogeneous solution, i.e., $y_c = c_1 Sin\,x + c_2 Cos\,x$, and y_p is a particular solution to D.E., this solution is of the form

$$y_p = u_1 Sin\,x + u_2\,Cos\,x$$

where u_1' and u_2' satisfy the system

$$Sin\,x\,u_1' + Cox\,x\,u_2' = 0$$

$$Cos\,x\,u_1' - Sin\,x\,u_2' = 1cos^3 x$$

Thus

$$u_1' = 0cosx$$

sin x cos x

= $1\cos^2 x$

and

$$u_2 = sinx0$$

sin x cos x

= -sin $xcos^3 x$

Thus

$$u_1 = \int dxcos^2 x = tan\,x$$

and

$$u_2 = -\int sinxcos^3 xdx = \int d(cosx)cos^3 x = (cosx)^{-3+1} - 3 + 1 = -12cos^2 x$$

The general solution to D.E. is

$$y = c_1 sinx + c_2 cosx + tanx\,sinx - 12cos^2 xcos\,x$$

8.

Solution. Example (similar). Find the general solution to D.E.

$$y'' + 4y = 1sin^3 2x, -n < \ x \ < 0.$$

9. Find the solution of the following initial value problem:

$$y' + 4x^3 y = x^3, \quad y(0) = 1$$

Solution. $Integratingfactor\mu(x) = e^{\int 4x^3 dx} = e^{x^4} Thus, \left(e^{x^4} y\right)' = x^3 e^{x^4}$

$$\Rightarrow e^{x^4} = \int x^3 e^{x^4} + c$$

$$\Rightarrow e^{x^4} y = 14e^{x^4} + c$$

$$\Rightarrow y(x) = 14 + ce^{-x^4}$$

$$1 = y(0) = 14 + c \ \Rightarrow c = 34$$

Therefore,

$$y(x) = 14 + 34e^{-x^4}$$

10. A certain population increases at a rate proportional to the cubic root of the population present. In the $t = 0$ the population is 1000. At $t = 3$ later it is 27,000. First find a formula that gives the population p(t) at any time and then compute the population at $t = 45/8$.

Solution.

$$dPdt = rp^1 3 \Rightarrow p^{-13} dP = rdt \Rightarrow p^- 13 + 1 - 13 + 1 = rt + c$$

$$p^2 3 = 23(rt + c) \cdot 1000^2 3 = 23c \Rightarrow 100 = 23c$$

$$\Rightarrow c = 150 \ \Rightarrow P^2 3 = 23(rt + 150)$$

$$27,000^2 3 = 23r \cdot 3 + 100 \Rightarrow 900 = 2r + 100 \Rightarrow \ r = 400$$

$$\Rightarrow P^2 3 = 23(400t + 150)$$

$$P(45/8) = 23(400 \cdot 458 + 150) = 1,600$$

$$P(t) = [23(400t + 150)]^3 2$$

$$P(458) = 1600^3 2 = 40^3 = 64,000$$

11.

A man with a parachute jumps from a great hight. The combined weigt of man and parachute is 160 pounds and the force of the air resistance is equal to $12v$ when the parachute is closed. After 10 seconds he opens his parachute. Find his speed $v(t)$ at any time before he opens his parachute and at the time he opens his parachute. It is given that $1 - e^{-1} 0.632$.

Solution.

$$mdvdt = mg - 12v$$

$$\Rightarrow dvdt + 12mv = g$$

$$m = wg = 16032 = 5$$

$$dvdt + 0.1v = 32$$

$$\Rightarrow \left(ve^{0.1t}\right)^1 = 32e^{0.1t}$$

$$\Rightarrow ve^{0.1t} = 320.1e^{0.1t} + c$$

$$\Rightarrow V(t) = 320 + ce^{-0.1t},\ 0 = v(0) = 320 + c$$

$$V(t) = 320\left(1 - e^{-0.1t}\right)$$

$$V(10) = (320)x(0.632) = 202.24\, ft./sec.$$

$$m \bullet t = 0$$

12. Find the general solution of the D.E.

$$y'' - 10y' + 25y = 0$$

Then find the solution with $y(0) = 1$ and $y'(0) = 9$.

Solution.

$$r^2 - 10r + 25 = (r - s)^2 = 0 \Rightarrow r = 5$$

$$y = c_1 e^{5x} + c_2 x e^{5x}$$

$$1 = y(0) = c_1$$

$$y' = 5c_1 e^{5x} + c_2 \left(5xe^{5x} + e^{5x}\right)$$

$$y = y(0) = 5c_1 + c_2 \Rightarrow c_2 = 4$$

So,

$$y = c_1 e^{5x} + c_2 x e^{5x}$$

$$y = e^{5x} + 4xe^{5x}$$

13. Find a particular solution to the D.E:

$$y'' + y = 1cosx$$

Then find the general solution to the above D.E.

Solution We have that $y_1 = cosx$ and $y_2 = sinx$ are solutions to $y'' + y = 0$. Then, $y_p = y_1 u_1 + y_2 u_2$ where

$$\{ y_{\,1}\, u_1' + y_2 u_2' = 0$$

$$u_1' = |0 sinx$$

$|cosxsinx$

= -sin xcos x $\Rightarrow u_1 = -\int sinxcosxdx = ln|cosx|$

$$u_2' = |cosx0$$

So,

$$y_p = cos\, ln|cos\, x| + x\, sin\, x$$

$$y = c_1 cos\, x + c_2\, sin\, x + y_p$$

13. Find a particular solution to the D.E:

$$y'' + y = 1cosx$$

Then find the general solution to the above D.E.

Solution We have that $y_1 = cosx$ and $y_2 = sinx$ are solutions to $y'' + y = 0$. Then, $y_p = y_1 u_1 + y_2 u_2$ where

$$\{ y_1 u_1' + y_2 u_2' = 0$$

$$u_1' = 0sinx$$

cos x sin x

= -sin xcos x $\Rightarrow u_1 = -\int sinxcosxdx = ln|cosx|$

$$u_2' = cosx0$$

14. Find the general solution to the differential equation

$$xy'' - (2x + 1)\, y' + (x + 1)\, y = 0$$

given that $y_1(x) = e^x$ is a solution.

Solution. We look for a second independent solution y_2 in the form

$$y_2 = y_1(x)u(x) = e^x u(x)$$

We must have

$$x\, (e^x u)'' - (2x + 1)\, (e^x u)' + (x + 1)\, e^x u = 0$$

or

$$x\left(e^{x}u'' + 2e^{x}u' + e^{x}u\right) - (2x+1)\left(e^{x}u' + e^{x}u\right) + (x+1)\,e^{x}u = 0$$

or

$$xu'' - u' = 0$$

Let $u = u'$. Then $xu' = u$ or $duu = dxx$ or $ln\, u = ln\, x$ or $u = x$ or $u' = x$ or $u' = 12x^2$. Thus, $y_2(x) = 12x^2e^x$ and the general solution is given by

$$y(x) = e^{x}\left(c_1 + c_2x^2\right).$$

15. $(Variation of parameters)\,.Find the general solution to the differential equations:$

$$i.\; y'' - y = 1e^{x} + 1$$

$$ii.\; y'' - 2y' + y = e^{x}\, ln\, xx^2$$

$$iii.\; xy'' - (2x+1)\,y' + (x+1)\,y = g(x),$$

given that $y_1 = e^x$ and $y_2 = x^22e^x$ are two independent solutions.