Mathematics 226.02: Differential Equations and Linear Algebra
Fall Semester 1997
Solutions of Exam 1

1. Solve the following initial value problem and determine the interval, where the solution is defined.

$$(x^2-4)\frac{dy}{dx}+2xy=\frac{2x^2-8}{x}\,, \qquad y(1)=2$$

Solution: This is a linear equation, which can be rewritten as

$$\frac{dy}{dx}+\frac{2x}{x^2-4}y=\frac{2}{x}.$$

The integrating factor is

$$\mu=\exp\int\frac{2x}{x^2-4}dx=x^2-4$$

and therefore the general solution is given by

$$y=\frac{\int(2x-\frac{8}{x})dx+c}{x^2-4}=\frac{x^2-8\ln x+c}{x^2-4}.$$

From the initial condition one calculates $c=-7$ so that the solution of the problem is

$$y=\frac{x^2-8\ln x-7}{x^2-4} \quad \text{defined on the interval } 0<x<2.$$

2. Find the value of a, for which the equation is exact, and then solve it using that value of a.

$$(y^2+x^3+2x)+(axy+y^2+3)\frac{dy}{dx}=0$$

Solution: The equation is exact for $a=2$.
For this value, the solution of the differential equation is obtained as follows: define

$$\psi=\int(y^2+x^3+2x)dx+h(y)=xy^2+\frac{1}{4}x^4+x^2+h(y).$$

Then the partial derivative with respect to y of ψ has to be equal to $2xy+y^2+3$, thus we get

$$\psi_y=2xy+h'(y)=2xy+y^2+3,$$

hence $h'(y)=y^2+3$ and we can take $h(y)=\frac{1}{3}y^3+3y$. The implicit formula for the solution is

$$xy^2+\frac{1}{4}x^4+x^2+\frac{1}{3}y^3+3y=c.$$

3. (a) Solve the differential equation

$$x^3 \frac{dy}{dx} + (y-2)^2 = 0$$

Solution: Separating variables leads to the expression

$$\frac{1}{(y-2)^2} dy + \frac{1}{x^3} dx = 0.$$

Integrating this equation yields

$$-\frac{1}{y-2} - \frac{1}{2x^2} = c$$

which can be solved for y:

$$y = \frac{2x^2}{-2cx^2 - 1} + 2.$$

The function $y = 2$ is a solution, too.

(b) Find an implicit expression for the solution of the differential equation

$$x\frac{dy}{dx} = y + 2xe^{\frac{y}{x}}$$

Solution: This is a homogeneous equation

$$\frac{dy}{dx} = \frac{y}{x} + 2e^{\frac{y}{x}}.$$

The transformation of variables $v = \frac{y}{x}$ leads to the equation

$$\frac{1}{x} dx = \frac{1}{2e^v} dv$$

which after integration yields

$$\ln|x| + c = -\frac{1}{2}e^{-v}$$

and substituting back $\frac{y}{x}$ for v results in

$$2\ln|x| + C + e^{\frac{-y}{x}} = 0.$$

4. (a) A skydiver weighing 160 lbs. (including equipment) falls vertically down from a certain high altitude. The parachute opens when the skydiver reaches speed equal to 130 ft/sec. Assume that $t = 0$ when the parachute opens and that the air resistance is $10|v|$ when the parachute is open. Assume also that the gravitational force is 32 ft/sec^2.

(i) Write the initial value problem (i. e. a differential equation and an initial condition) for the speed $v(t)$ at any time $t > 0$. **DO NOT SOLVE IT!**

(ii) Find the limiting velocity v_L of the skydiver after the parachute opens.

Solution: Newton's law says: mass $\times$ acceleration = external force.
In the present case the external force is given by

$$\text{weight of the skydiver minus air resistance } = 160 - 10v$$

(the weight acts in the direction of the motion, the air resistance acts in the opposite direction). The mass m of the skydiver is related to his/her weight via

$$\text{weight } = 160 = m \times \text{gravitational force, thus } m = 160/32 = 5.$$

Therefore we get the equation $5\frac{dv}{dt} = 160 - 10v$. The motion under consideration starts with time $t = 0$ when the parachute opens, which is when the speed is 130. Hence we get the initial value problem

$$\frac{dv}{dt} = 32 - 2v, \qquad v(0) = 130.$$

The limiting velocity is reached when $v' = 0$, thus we have to solve the equation $32 - 2v = 0$, which leads $v_L = 16$.

(b) Circle the differential equation whose direction field is shown in the following picture.

A. $y' = y(x + y)$ B. $y' = (x - y)(x + y)$ C. $y' = (2 - y)(x + y)$

D. $y' = (y - 2)(x + y)$ E. $(x - 2)(x + y)$

5. (a) Consider a tank holding initially 500 gallons of a salt solution with concentration 0.3 lb of salt per gallon. A solution containing 0.4 lb of salt per gallon is pumped into the tank at a rate of 3 gallons per minute, and the well-stirred mixture flows out of the tank at a rate of 5 gallons per minute. Write the initial value problem (i. e. a differential equation and an initial condition) needed to find the amount $S(t)$ of salt in the tank at time $t > 0$ prior to the instant when the tank is empty. **DO NOT SOLVE IT!**

Solution: Measure the time in minutes.
Each minute $3 \times 0.4 = 1.2$ pounds of salt flow into the tank.
Each minute 3 gallons of solution enter the tank whereas 5 gallons flow out. As a result, the tank looses 2 gallons per minute. Hence , at time t there are $500 - 2t$ gallons of solution in the tank.
The amount of *salt per gallon* in the tank at time t is therefore given by the ratio $\frac{S(t)}{500-2t}$.
This means that $5 \times \frac{S(t)}{500-2t}$ pounds of salt are leaving the tank at time t.
At the beginning $500 \times 0.3 = 150$ pounds of salt are in the tank.
All this together leads to the initial value problem

$$\frac{dS}{dt} = 1.2 - \frac{5}{500 - 2t} S(t), \quad S(0) = 150.$$

(b) Find the constant (equilibrium) solutions of the differential equation

$$\frac{dy}{dt} = -0.002(1 - \frac{y}{3})(1 - \frac{y}{7})y$$

and classify each one as asymptotically stable or unstable. Sketch the graphs of the four solutions y_1, y_2, y_3, y_4, which have initial values $y_1(0) = 3$, $y_2(0) = 4$, $y_3(0) = 7$, $y_4(0) = 1$.

Solution: The solutions $y = 0$ and $y = 7$ are asymptotically stable, the solution $y = 3$ is unstable.