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Math 226: Differential Equations and Linear Algebra
Fall Semester 1998
Final Exam
December 17, 1998

This Examination consists of two sections: Section I has 8 multiple choice problems worth 6 points each. Your answers are to be **x**-ed on the cover sheet. Section II has 9 partial credit problems. For maximum credit **show all your work**. Put your answers on the answer line if provided or circle it. Cross out all work you don't want graded.
This booklet consists of 12 sheets of paper including the front cover (not counting 2 blank pages at the end).
Calculators, books, and notes are not allowed.

Multiple Choice Problems

1.	(a)	(b)	(c)	(d)	(e)	**5.**	(a)	(b)	(c)	(d)	(e)
2.	(a)	(b)	(c)	(d)	(e)	**6.**	(a)	(b)	(c)	(d)	(e)
3.	(a)	(b)	(c)	(d)	(e)	**7.**	(a)	(b)	(c)	(d)	(e)
4.	(a)	(b)	(c)	(d)	(e)	**8.**	(a)	(b)	(c)	(d)	(e)

Scores

1 – 8	9	10	11	12	13	14	15	16	17	**Total**
48	12	10	10	10	10	10	15	10	15	150

Sign the pledge:
"On my honor, I have neither given nor received unauthorized aid on this Exam."

Signature: ______________________________

GOOD LUCK

1. Find the differential equation whose direction field is shown in the following picture.

(a) $y' = y(2x - y)$

(b) $y' = x(x + 2y)$

(c) $y' = x(2x + y)$

(d) $y' = y(x - 2y)$

(e) $y' = x(2x - y)$

2. Which of the following is the solution of the initial value problem

$$ty' + (t + 1)y = 1, \quad y(1) = 0.$$

(a) $y = \dfrac{1 - e^{-t}}{t}$

(b) $y = \dfrac{1 + ce^{-t}}{t}$

(c) $y = e^{1-t} - 1$

(d) $y = \dfrac{1 - e^{1-t}}{t}$

(e) $y = t(1 - e^{1-t})$

3. The following differential equation

$$xy' = 2y + xe^{\frac{y}{x}}$$

(a) is linear

(b) is separable

(c) is homogeneous

(d) has no solution passing through the point $(1, 2)$

(e) satisfies none of the above

4. Which of the following differential equations is not guaranteed to have a solution $y(x)$ for $x \in (-1, 1)$ such that $y(0) = 1,\ y'(0) = 1$.

(a) $y'' - 2y' + y = x$ **(b)** $xy'' - y' + 10y = 0$ **(c)** $y'' - \dfrac{y'}{(x-1)^2} + e^x y = 0$

(d) $y'' - xy' + x^2 y = x^3$ **(e)** $e^x y'' + (\sin x)y' + (\cos x)y = \tan x$

5. The system

$$\begin{bmatrix} 1 & 3 & -2 \\ -1 & 2 & -4 \\ 2 & -1 & 3 \end{bmatrix} \begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix} = \begin{pmatrix} 7 \\ 3 \\ 4 \end{pmatrix}$$

has

(a) no solution

(b) the solution $(0, 0, 0)$

(c) exactly one solution

(d) the solution $(1, 2, 0)$

(e) infintely many solutions

6. What is the determinant of

$$A = \begin{bmatrix} 1 & 1 & 1 & 3 \\ 1 & 2 & 3 & 4 \\ 0 & 1 & 2 & 0 \\ 0 & -2 & -4 & -1 \end{bmatrix}$$

(a) -2

(b) -1

(c) 0

(d) 1

(e) 2

7. Which of the following is **not** a subspace?

(a) $\{(x, y, z) \in \mathbb{R}^3 \,|\, x + y + z = 0\}$

(b) The set of functions $f : \mathbb{R} \to \mathbb{R}$ such that $f(x + 2\pi) = f(x)$

(c) $\left\{ \begin{pmatrix} x \\ y \end{pmatrix} \in \mathbb{R}^2 \,\middle|\, \begin{bmatrix} 1 & 1 \\ 0 & 1 \end{bmatrix} \begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix} \right\}$

(d) $\{$differentiable functions $f : \mathbb{R} \to \mathbb{R}$ such that $f''(x) = 0\}$

(e) $\{(v_1, v_2, v_3) \in \mathbb{R}^3 \,|\, (v_1, v_2, v_3) \cdot (1, 2, 3) = 1\}$

8. Which of the following transformations is **not** linear?

(a) $L : \mathbb{R}^3 \to \mathbb{R}^2, \quad L(x, y, z) = (x, z)$

(b) $L : \mathbb{R}^4 \to \mathbb{R}^4, \quad L(v) = \mathrm{Pr}\,[v, (1, 1, 1, 1)]$

(c) $L : \mathbb{R}^2 \to \mathbb{R}, \quad L(v) = v \cdot v$

(d) $L(f) = f' + f$ (f a differentiable function)

(e) $L : \mathbb{R}^2 \to \mathbb{R}^2, \quad L(\begin{pmatrix} x \\ y \end{pmatrix}) = \begin{pmatrix} x \\ y \end{pmatrix} - \begin{bmatrix} 1 & -1 \\ 0 & 3 \end{bmatrix} \begin{pmatrix} x \\ y \end{pmatrix}$

9. **(a)** Find the general solution of

$$y'' + 4y' + 4y = 0.$$

Answer: ______________________________

(b) Find the general solution of

$$y'' + 4y' + 4y = 5\sin x + 18e^x.$$

Answer: ______________________________

10. Given the differential equation

$$y' = 2y^2(4 - y^2)$$

(a) Find the equilibrium solutions and classify each one as asymptotically stable, unstable, or semistable.

Answer: ____

(b) Sketch the solutions y_1, y_2, y_3, and y_4 satisfying the initial conditions $y_1(0) = 1$, $y_2(0) = 2$, $y_3(0) = 3$, and $y_4(0) = -1$. Label each function in your picture.

11. One solution of

$$(1-x)y'' + xy' - y = 0$$

is given by $y = e^x$. If we want to find the general solution by setting $y = v(x)e^x$, what differential equation must v satisfy? **Do not solve that differential equation for v.**

Answer: ____

12. Suppose we want to express the solution of the initial value problem

$$y'' - 2xy = 0, \quad y(0) = 2, \quad y'(0) = -1$$

using a power series centered at $x = 0$. Compute the first four **non-zero** terms of this power series.

Answer: ____

13. Find the general solution in implicit form of the differential equation

$$(y \sin x - 2xe^y) + (2 - \cos x - x^2 e^y)y' = 0$$

Answer: ________________________________

14. Find the inverse of

$$A = \begin{bmatrix} 1 & 2 & 3 \\ 2 & 4 & 5 \\ 3 & 5 & 6 \end{bmatrix}$$

Answer: ________________________________

15. Given the following matrix in row echelon form and the vector b.

$$A = \begin{bmatrix} 1 & 3 & 4 & -3 & 1 \\ 0 & 1 & 2 & -1 & 2 \\ 0 & 0 & 0 & 1 & -5 \\ 0 & 0 & 0 & 0 & 0 \end{bmatrix}, \quad b = \begin{pmatrix} 3 \\ 1 \\ -4 \\ 0 \end{pmatrix}$$

(a) Find the general solution of the inhomogeneous equation $Ax = b$.

(b) Find a basis for the solution space of the homogeneous system $Ax = 0$. [Hint: Look at your work in (a).]

(c) What are the dimensions of the rowspace of A and of the columnspace of A?

16. Given the vectors

$$v_1 = (1, 0, 1, 0),\ \ v_2 = (0, 0, 2, 2),\ \ v_3 = (3, 4, 3, -12) \in \mathbb{R}^4.$$

(a) Find an orthogonal basis w_1, w_2, and w_3 for the subspace $\operatorname{span}\{v_1, v_2, v_3\}$.
[Note: The set $\{w_1, w_2, w_3\}$ need not be orthonormal.]

Answer: ______________________________

(b) Express v_3 as a linear combination of w_1, w_2, and w_3.

Answer: ______________________________

17. (a) Find all eigenvalues of the matrix

$$A = \begin{bmatrix} 5 & -4 \\ 6 & -5 \end{bmatrix}$$

Answer: ____

(b) Find all eigenvectors belonging to each eigenvalue of A.

Answer: ____

(c) Is A diagonalizable? If not, why not? If so, find matrices P and Λ such that $P^{-1}AP = \Lambda$ and Λ is diagonal.