Math 228, Final Exam

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Instructions: There are 17 problems with a total of 150 possible points. The point-value of each problem is indicated. Show your work clearly on all problems, except on the true/false problems and where noted.
You have two hours for the exam.
Calculators, notes, and books are prohibited.
This exam is bound by the provisions of the Notre Dame Honor Code.

Name:

1. (8 points) If

det [
[c]ccc
a b c
m n p
x y z
= 3,

what is the value of the determinant

det [
[c]ccc
3a 3b 3c
m+a n+b p+c
-x -y -z

?

2. (8 points) Suppose that the population $P(t)$ of bacteria in a certain culture satisfies a differential equation of the form

dP $\overline{dt=kP}$

for some constant k. Suppose that the initial population of bacteria is 10,000 and that the population after one day is 30,000. Determine the constant k and find the formula for $P(t)$. How long will it take the population of bacteria to reach 1,000,000?

Note: Your answer will involve logarithms, which you do not have to evaluate numerically.

3. (8 points.) Let A be the matrix

[

[c]rrr

2 3 -1

4 -1 -2

-2 -1 1

.

a) Solve the system

Ax=[

[c]r

-5

4

1

.

b) Find a basis for $NS(A)$.

c) Is zero an eigenvalue of A? Why or why not?

4. (8 points) Find the solution of the equation

y$'' - 4y' + 4y = 0$

subject to the initial conditions

y$(0) = 0$

y$'\,(0) = 1$.

5. (10 points) a) Compute the inverse of the matrix

A=[

[c]rrr

2 1 0

-4 -1 -3

4 1 2

.

b) Find the L-U decomposition of A. No row interchanges should be necessary.

6. (8 points) Solve the differential equation

dy$_{\overline{dx=y(x+1)}}$

subject to the initial condition

y(1) = 2.

7. (10 points) a) Let

$p_1(x) = x^2 - x + 1$

$p_2(x) = x + 2$

$p_3(x) = 2x^2 - 3x.$

Is the set $S = \{p_1, p_2, p_3\}$ a basis for $\mathcal{P}$
$_2$?
b) Let $S = \{(1,0,0), (2,0,0), ($
0,1,1, $(2,2,2), (0,3,3)$. Find
a basis for the span of S.

8. (8 points) Find the general solution, in a form involving only real numbers, of the equation

$$y'' + y' + y = 0.$$

Describe the behavior of the system as t increases.

9. (10 points) In each of the following, you are given a vector space V and a subset S of V. In each case, state whether or not S is a subspace of V. If so, state the dimension of S. (No work is required; credit will be assigned strictly on your answers.)

a) $V = \mathbb{R}^3$, $S = \{(x, y, z) \,|\, z = x + y\}$

b) $V = \mathbb{R}^3$, $S = \{(x, y, z) \,|\, z = x + y + 1\}$

c) $V = \mathbb{R}^3$, $S = \{(0, t + 1, 2t + 2) \,|\, t \in$
R

d) $V = C\,[0, 2]\,,\ S = \{f \in V |\, f$
degree at most 2

e) $V = \mathcal{P}_3$, $S = \Big\{p \in V \,\Big|\, \int_0^1 p\,($
x $dx = 1$

10. (8 points) For each of the following, determine whether or not the statement is always true.

a) If Q is an orthogonal $n \times n$ matrix, then Q^T is orthogonal.

b) If Q is an orthogonal $n \times n$ matrix, then Q^2 is orthogonal.

c) If A is an $n \times n$ matrix and v is an eigenvector for A with eigenvalue λ, then $3v$ is an eigenvector for A with eigenvalue 3λ.

d) If A and B are symmetric $n \times n$ matrices, then AB is symmetric.

11. (8 points) Find the general solution of the equation

dy$\frac{}{dt-\frac{1}{t}y=t^2.}$

12. (8 points) What is the largest interval on which the solution of

$$\sqrt{2-t^2}\,\frac{dy}{dt} + e^t\,y = t^3$$

y(0) = 1

is guaranteed to exist?

13. (10 points) Determine whether each of the following two differentials is exact. In each case, solve the equation, using an integrating factor if necessary.

[c]cccc

a) y dx+$(x + 6y)\ dy = 0$

b) $2\sin y\ dx + x\cos y\ dy = 0$

14. (8 points) Suppose that A is a 3×3 matrix whose characteristic polynomial is $(\lambda + 1)\,(\lambda + 2)\,($
$\lambda + 5$. For each of the following statements, determine whether or not it is always true.

a) A is invertible.

b) A is diagonalizable.

c) There exist 3 linearly independent eigenvectors for A.

d) If v_1 is an eigenvector for A with eigenvalue -1 and v_2 is an eigenvector for A with eigenvalue -5, then v_1 and v_2 are orthogonal.

15. (10 points) Find the general solution of the equation

$$\mathrm{y}'' - y' - 2y = t + \sin t.$$

16. (10 points) Find an orthonormal basis for $\mathbb{R}^3$ consisting of eigenvectors for the matrix

A=[

[c]rrr

1 -2 -2

-2 1 2

-2 2 1

.

Hint: the characteristic polynomial of A is $(\lambda$

$+1^2\,(\lambda - 5)\,.$

17. (10 points) Suppose that A is an arbitrary 2×2 matrix whose characteristic polynomial is $\lambda^2 + 5\lambda + 4$. What is the characteristic polynomial of A^2? Explain your reasoning.