Name: ______________________________

Instructor: ______________________________

MATH 228: Introduction to Linear Algebra and Differential Equations

Exam II

April 11, 2003

There are 20 problems on pages 2–11. Each problem is worth 5 points. You must work on the exam entirely on your own. You are not allowed to discuss the exam with anyone until after they are turned in. You may use your textbook and a calculator. *To receive credit, you must show all your work and include all important steps and explanations. A correct answer without supporting work will receive no credit.*

Please work out the problems on separate sheets and then copy your work and answers neatly into the space below the appropriate problem. You will lose credit if your answers are difficult to read or your work is poorly organized.

	Score
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1. Let V be the vector space $C[0, 2\pi]$ with inner product defined by $\langle f, g\rangle = \int_0^{2\pi} f(x)g(x)\,dx$. Use the Gram-Schmidt process to find an orthonormal basis for the subspace of V spanned by 1, $\sin(x)$, x.

2. Let $W = \text{Span}\{(1,2,3,4),(1,1,1,1),(1,4,7,10)\}$. Find a basis for $W^{\perp}$.

3. Find the QR-decomposition of $\begin{bmatrix} 1 & -2 & 0 \\ -2 & 1 & 1 \\ 2 & 2 & 1 \end{bmatrix}$

4. Let $A = \begin{bmatrix} 1 & 0 & 1 \\ 0 & 1 & -1 \\ -1 & 1 & 1 \\ 1 & -1 & 0 \end{bmatrix}$ and $\mathbf{b} = \begin{bmatrix} 1 \\ 1 \\ 15 \\ 0 \end{bmatrix}$. Determine the least squares solution of the linear system $A\mathbf{x} = \mathbf{b}$, as well as the projection of $\mathbf{b}$ on the column space of A.

5. Let $A = \begin{bmatrix} -2 & 3 & 3 \\ -1 & 5 & 6 \\ 1 & -4 & -5 \end{bmatrix}$. Find the eigenvalues and bases for the eigenspaces of A^{20}.

6. Let A be a 6×6 matrix with characteristic equation $\lambda^3(\lambda+2)^2(\lambda-5) = 0$. Give the possible dimensions for the eigenspaces of A.

7. Let $A = \begin{bmatrix} 5 & -1 & 1 \\ -1 & 5 & 1 \\ 1 & 1 & 5 \end{bmatrix}$. Find a matrix P that orthogonally diagonalizes A and determine $P^{-1}AP$.

8. Consider the linear transformation $T : P_2 \to P_4$ defined by $T(p(x)) = x^2(p(x) - p(0))$. Determine a basis for the range, a basis for the kernel, the rank and the nullity of T.

9. Let $T : \mathbb{R}^3 \to \mathbb{R}^2$ be the linear transformation $T(x, y, z) = (x + y - 2z, 3x - 2y + z)$. Find the matrix $[T]_{B',B}$ where $B = \{(1,0,1),(0,1,-1),(1,-1,1)\}$ and $B' = \{(1,2),(2,5)\}$.

10. Consider $V = \text{Span}\{e^x, xe^x, x^2e^x\}$, a subspace of the vector space of differentiable functions. Let $T : V \to V$ be the linear operator defined by $T(f(x)) = f''(x) + f'(x) - f(x)$. Find the matrix $[T]_B$ where $B = \{e^x, xe^x, x^2e^x\}$, and use this matrix to solve $T(f(x)) = x^2e^x$ for $f(x)$.

11. Determine the values of r for which the differential equation $6t^2y'' + ty' + y = 0$ has a solution of the form $y = t^r$.

12. Find the general solution of the differential equation $ty' + 2y = \sin(t)$.

13. Solve the initial value problem $y' = \dfrac{xy^2 + xy}{x + 1}$, $y(0) = 1$, in explicit form.

14. A 1000 gallon tank contains 500 gallons of water mixed with 10 pounds of impurities. Pure water is added to the tank at a rate of 5 gallons per minute. At the same time the water in the tank is being filtered. The filtering process takes 10 gallons per minute from the tank, removes 50% of the impurities, and returns the filtered water to the tank at the same rate. The tank is kept well mixed. Determine the concentration of impurities when the tank is filled to capacity.

15. Suppose a population N satisfies the logistic equation $\dfrac{dN}{dt} = N(N^2 - 100)$. Without solving the equation, determine the limiting value of the population for each of the following initial values:
a) $N(0) = 5$ *b)* $N(0) = 10$ *c)* $N(0) = 15$

16. Find an implicit solution to the differential equation

$$(e^{xy}y + 3x^2y - y^2)dx + (e^{xy}x + x^3 - 2xy - 1)dy = 0$$

that satisfies $y(0) = 2$.

17. Find an integrating factor for the differential equation $(y + (y - 2x)e^{-x})\,dx + (1 + xe^{-x})\,dy = 0$.

18. Find the solution of the initial value problem $2y'' - y' - y = 0$, $y(0) = 1$, $y'(0) = 0$.

19. Determine the longest interval in which the initial value problem $(t+1)y'' + \sqrt{t+2}y' + y = \tan(t)$, $y(0) = 0$, $y'(0) = 1$, is certain to have a unique twice differentiable solution.

20. Determine which of the following pairs of functions are linearly independent.

a) $\cos(x)$, $\sin(x)$

b) $\dfrac{x}{1-x}$, $1 - \dfrac{1}{1-x}$

c) e^x, e^{2x}

d) x, $x - 1$

e) $\ln(x)$, $\ln(x^2)$