Name:______________________________

Instructor:______________________________

**MATH 228: Introduction to Linear Algebra and Differential Equations**

**Final Exam**

*May 7, 2003*

Record your answers by placing an $\times$ through one letter for each problem on this page. There are 21 questions worth 6 points each. You start with 24 points.

You may *not* use a calculator.

Determine the number of parameters in the general solution of the linear system

$$\begin{array}{rcrcrcrcl} x & + & y & + & z & + & w & = & 0 \\ x & + & 2y & & & & & = & 1 \\ & & y & - & z & - & w & = & 1 \end{array}$$

2 0 1 3 none, it is inconsistent

Compute the determinant of $\begin{bmatrix} 0 & 1 & 2 & 3 \\ 1 & 2 & 3 & 0 \\ 2 & 3 & 0 & 1 \\ 3 & 0 & 1 & 2 \end{bmatrix}$.

96 $-16$ 0 48 $-72$

Let $T$ be a linear transformation $T : \mathbf{R}^3 \to \mathbf{R}^3$ that satisfies $T(1,0,1) = (0,1,0)$, $T(0,1,1) = (1,0,0)$, and $T(1,1,0) = (0,0,1)$. Find $T^{-1}(1,2,3)$. $(5,4,3)$ $(3,4,5)$ $(4,5,3)$ $(3,5,4)$ $(4,3,5)$

Find the first coordinate of the vector $(5,6,7)$ with respect to the basis $\{(1,0,1),(1,0,2),(0,-1,0)\}$. 3 2 $-6$ $-1$ 5

Determine which of the following sets is *not* subspace of the given vector space $V$. All $A$ in $M_{n,n}$ such that $A^{-1} = -A$. All polynomials $a_0 + a_1x + a_2x^2 + a_3x^3$ in $P_3$ such that $a_3 = a_0 + a_1 + a_2$. All $A$ in $M_{n,n}$ such that $A^T = -A$. All functions $f(x)$ in $C[0,1]$ such that $f(1) = 0$. All functions $f(x)$ in $C[0,1]$ such that $\int_0^1 f(x)dx = 0$.

Determine which of the following statements is true about the functions $\sin(x)$, $\cos(-x)$, $\sin(x - \pi/2)$.

They are linearly dependent since their span is two-dimensional. They are linearly independent because $\sin(x)$ is not a multiple of $\cos(-x)$. They are linearly independent by the Wronskian test. They are linearly dependent since $\cos(-x) = \cos(x)$. They are linearly dependent since their kernel is non-zero.

Find a basis for the column space of the matrix $\begin{bmatrix} 1 & -1 & 2 & 1 \\ -1 & 1 & -1 & 0 \\ 2 & -2 & 1 & -1 \end{bmatrix}$.

$\begin{bmatrix} 1 \\ -1 \\ 2 \end{bmatrix}, \begin{bmatrix} 2 \\ -1 \\ 1 \end{bmatrix}$ $\begin{bmatrix} 1 \\ -1 \\ 2 \end{bmatrix}, \begin{bmatrix} 2 \\ -1 \\ 1 \end{bmatrix}, \begin{bmatrix} 1 \\ 0 \\ -1 \end{bmatrix}$ $\begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix}, \begin{bmatrix} 2 \\ 1 \\ 0 \end{bmatrix}$ $\begin{bmatrix} 1 \\ -1 \\ 2 \end{bmatrix}, \begin{bmatrix} -1 \\ 1 \\ 1 \end{bmatrix}$ $\begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix}, \begin{bmatrix} 2 \\ 1 \\ 0 \end{bmatrix}, \begin{bmatrix} 1 \\ 1 \\ -3 \end{bmatrix}$

Determine the rank and nullity of the matrix $A = \begin{bmatrix} 1 & 2 & 3 & 4 \\ 0 & 2 & 3 & 4 \\ 0 & 0 & 3 & 4 \\ 1 & 2 & 3 & 4 \end{bmatrix}$.

$\operatorname{rank}(A) = 3$, $\operatorname{nullity}(A) = 1$ $\operatorname{rank}(A) = 1$, $\operatorname{nullity}(A) = 3$ $\operatorname{rank}(A) = 2$, $\operatorname{nullity}(A) = 2$ $\operatorname{rank}(A) = 3$, $\operatorname{nullity}(A) = 3$ $\operatorname{rank}(A) = 1$, $\operatorname{nullity}(A) = 1$

A symmetric $3 \times 3$ matrix $A$ has characteristic equation $(\lambda - 1)(\lambda - 2)^2 = 0$. Determine which of the following statements is true.

$A$ is orthogonally diagonalizable.

The $\lambda = 1$ eigenspace has dimension 1 but the $\lambda = 2$ eigenspace could have dimension 1 or 2. $A$ is diagonalizable but may not necessarily be orthogonally diagonalizable. $A$ is not necessarily diagonalizable. $A$ must be a diagonal matrix.

A $3 \times 3$ matrix $A$ has eigenvalues $\lambda = 1$ and $\lambda = 2$. A basis for the eigenspace corresponding to $\lambda = 1$ is $\begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix}$. A basis for the eigenspace corresponding to $\lambda = 2$ is $\begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix}, \begin{bmatrix} 1 \\ -1 \\ 0 \end{bmatrix}$. Determine which of the following is the matrix $A$.

$\begin{bmatrix} 1 & -1 & 2 \\ 0 & 2 & 0 \\ 0 & 0 & 2 \end{bmatrix}$ $\begin{bmatrix} 1 & 0 & 0 \\ 0 & 2 & 0 \\ 0 & 0 & 2 \end{bmatrix}$ $\begin{bmatrix} 1 & 0 & 0 \\ -1 & 2 & 0 \\ 2 & 0 & 2 \end{bmatrix}$ $\begin{bmatrix} 1 & 1 & 0 \\ 0 & 2 & 0 \\ 0 & 0 & 2 \end{bmatrix}$ $\begin{bmatrix} 1 & 1 & -2 \\ 0 & 0 & 1 \\ 0 & 1 & -1 \end{bmatrix}$

Find the standard matrix for the orthogonal projection of $\mathbf{R}^3$ onto the subspace spanned by $\begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix}$ and $\begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix}$.

$\frac{1}{2}\begin{bmatrix} 2 & 0 & 0 \\ 0 & 1 & 1 \\ 0 & 1 & 1 \end{bmatrix}$ $\frac{1}{2}\begin{bmatrix} 2 & 0 & 0 \\ 0 & 1 & 1 \\ 0 & -1 & 1 \end{bmatrix}$ $\begin{bmatrix} 1 & 1 & 0 \\ 1 & 0 & 1 \\ 1 & 0 & -1 \end{bmatrix}$ $\begin{bmatrix} 1 & 1 & -1 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$ $\begin{bmatrix} 1 & 1 & -1 \\ 0 & 1 & 0 \\ 0 & 1 & 0 \end{bmatrix}$

Determine which of the following is an orthogonal matrix.

$\begin{bmatrix} 0 & 0 & 0 & 1 \\ 0 & 0 & 1 & 0 \\ 0 & 1 & 0 & 0 \\ 1 & 0 & 0 & 0 \end{bmatrix}$ $\begin{bmatrix} 1 & -1 \\ 0 & 1 \end{bmatrix}$ $\begin{bmatrix} 1 & -1/\sqrt{2} & 1/\sqrt{3} \\ 0 & 1/\sqrt{2} & 1/\sqrt{3} \\ 0 & 0 & 1/\sqrt{3} \end{bmatrix}$ $\begin{bmatrix} 1 & 0 & 1 & 0 \\ 0 & 1 & 0 & 1 \\ 1 & 0 & -1 & 0 \\ 0 & 1 & 0 & -1 \end{bmatrix}$ $\begin{bmatrix} 0 & 2/\sqrt{6} & -1/\sqrt{3} \\ 1 & 1/\sqrt{6} & 1/\sqrt{3} \\ -1 & 1/\sqrt{6} & 1/\sqrt{3} \end{bmatrix}$

A linear transformation $T : P_2 \to P_2$ is given by $T(p(x)) = p(x+1) - p(x)$. Find the standard matrix $[T]_B$ for $T$ with respect to the basis B=$\{1, x, x^2 - 5\}$ of $P_2$.

$\begin{bmatrix} 0 & 1 & 1 \\ 0 & 0 & 2 \\ 0 & 0 & 0 \end{bmatrix}$ $\begin{bmatrix} 1 & 1 & 1 \\ 0 & 0 & 2 \\ 0 & 0 & 0 \end{bmatrix}$ $\begin{bmatrix} 0 & 0 & 0 \\ 1 & 0 & 0 \\ 1 & 2 & 0 \end{bmatrix}$ $\begin{bmatrix} 1 & 1 & 0 \\ 0 & 2 & 0 \\ 0 & 0 & 0 \end{bmatrix}$ $\begin{bmatrix} 0 & 0 & 0 \\ 0 & 1 & 1 \\ 0 & 0 & 2 \end{bmatrix}$

A linear transformation $T : M_{2,2} \to R^2$ has standard matrix $[T]_{B',B} = \begin{bmatrix} 1 & 3 & 1 & -1 \\ 0 & 0 & 2 & 4 \end{bmatrix}$ relative to the bases $B = \left\{ \begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix}, \begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix}, \begin{bmatrix} 0 & 0 \\ 1 & 0 \end{bmatrix}, \begin{bmatrix} 0 & 0 \\ 0 & 1 \end{bmatrix} \right\}$ of $M_{2,2}$ and $B' = \{(1,0),(0,1)\}$ of $\mathbf{R}^2$.

Determine which of the following matrices $A$ satisfy $T(A) = \begin{bmatrix} 1 \\ 2 \end{bmatrix}$.

$\begin{bmatrix} 0 & 1 \\ -1 & 1 \end{bmatrix}$ $\begin{bmatrix} 1 & 0 \\ 1 & 0 \end{bmatrix}$ $\begin{bmatrix} 0 & 1 \\ 1 & -1 \end{bmatrix}$ $\begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix}$ $\begin{bmatrix} 0 & 0 \\ 1 & -1 \end{bmatrix}$

Suppose a man suspended from a bungee cord stretches the cord 50 feet longer than its natural length of 100 ft. The man jumps from a bridge while attached to this cord and arrives at a distance of 150 ft below the bridge with a downward velocity of 40 ft/sec. Assume the bungee cord acts like a spring (i.e., it obeys Hooke's Law) and there are no frictional forces. Determine how far below the bridge the man will fall before springing up again. (The gravitational constant is $g = 32$ ft/sec$^2$.)

200 ft 220 ft 180 ft 260 ft 240 ft

The water bath surrounding a certain nuclear reactor contains radioactive material that decays with a half-life of 4 days. New radioactive material is added to the bath at a constant rate of 3 grams per day. Determine the amount in grams of radioactive material in the bath after many days.

$\frac{12}{\ln 2}$ $\frac{3}{4}$ $\frac{4 \ln 2}{3}$ $\frac{4}{3}$ 3

The differential equation $(x^2 - 6xy^2 + y^3)dx + (xy^2 - 3x^2y)dy = 0$

is exact. is homogeneous. has an integrating factor that is a function of $x$ alone. has an integrating factor that is a function of $y$ alone. *none of the above*

Solve the initial value problem $y'' + y' = 0$, $y(0) = 1$, $y'(0) = 0$.

$y = 1$ $y = e^{-t} + t$ $y = \cos(t)$ $y = (e^t + e^{-t})/2$ $y = e^t - t$

Let $V$ be the vector space of infinitely differentiable functions on $\mathbf{R}$. Find a basis for the kernel of the linear operator $L : V \to V$ defined by $L[y] = 9y'' + 6y' + y$.

$\{e^{-t/3}, te^{-t/3}\}$ $\{e^{t/3}, e^{-t/3}\}$ $\{t, e^{-3t}\}$ $\{t, e^{t/3}\}$ $\{e^{3t}, te^{3t}\}$

Find the form of a particular solution $y_p$ of $y'' - 2y' + 2y = e^t \cos(t) + e^{-t} \sin(t)$.

$y_p = te^t\big(A\sin(t) + B\cos(t)\big) + e^{-t}\big(C\sin(t) + D\cos(t)\big)$ $y_p = e^t\big(A\sin(t) + B\cos(t)\big) + e^{-t}\big(C\sin(t) + D\cos(t)\big)$ $y_p = e^t\big(A\sin(t) + B\cos(t)\big) + te^{-t}\big(C\sin(t) + D\cos(t)\big)$ $y_p = te^t\big(A\sin(t) + B\cos(t)\big) + te^{-t}\big(C\sin(t) + D\cos(t)\big)$ $y_p = Ate^t\sin(t) + Bte^{-t}\cos(t)$

Given that $y_1 = t$ and $y_2 = t^2$ are solutions of $y'' + p(t)y' + q(t)y = 0$ for $t > 0$, find a particular solution $y_p$ of $y'' + p(t)y' + q(t)y = t^3$.

$y_p = \frac{1}{12}t^5$ $y_p = -\frac{1}{4}t^4 + \frac{1}{3}t^3$ $y_p = t^{-1} - \frac{1}{3}t^{-3}$ $y_p = 4t^5 - 3t^4 + t^3$ $y_p = \frac{1}{6}t^3 - t^{-1}$