MATH 228: Intro to Linear Alg and Diff Eqns Name:______________________________

Exam II *April 8, 2004* Instructor:______________________________

There are 13 problems on 7 pages worth a total of 90 points. You start with 10 points.

The first 10 problems are multiple choice questions worth 6 points each. Record your answers to these problems by placing an $\times$ through one letter for each problem on this page.

Problems 11–13 are partial credit problems worth 10 points each. *You must show your work and all important steps to receive credit for these problems.*

Calculators are *not* allowed.

Let V be the vector space of continuous functions on $[0, 1]$ with an inner product defined by

$$\langle f(x), g(x)\rangle = \int_0^1 f(x)g(x)\,dx$$

Compute the projection of $f(x) = 4x^2 + 3x + 2$ on $g(x) = x$.

$9x$

$3x$

$x/3$

0

$x/9$

Find the least squares solution to the linear system

$$\begin{aligned} x - y &= 4 \\ 2x + y &= -1 \\ x + 2y &= 1 \end{aligned}$$

$x = 1$, $y = -1$

$x = 6$, $y = -3$

$x = -5$, $y = -9$

$x = 3$, $y = 1$

$x = 3$, $y = 6$

Determine a basis for the range of the linear transformation

$$T(x_1, x_2, x_3, x_4) = (x_1 + 2x_2 + x_3 + x_4,\ 2x_1 + 4x_2 + x_3 + 3x_4,\ 3x_1 + 6x_2 + 2x_3 + 4x_4)$$

$\{(1, 2, 3), (1, 1, 2)\}$

$\{(1, 0, 0), (1, -1, 0)\}$

$\{(1, 2, 1), (0, 0, 1)\}$

$\{(1, 2, 1), (2, 4, 1)\}$

$\{(1,2,3),(2,4,6)\}$

The matrix $A = \begin{bmatrix} 2 & 4 & 0 \\ 4 & 8 & 0 \\ 0 & 0 & 5 \end{bmatrix}$ has eigenvalues $\lambda_1 = 0$, $\lambda_2 = 5$, and $\lambda_3 = 10$. Find an orthogonal matrix B so that $B^{-1}AB = \begin{bmatrix} 0 & 0 & 0 \\ 0 & 5 & 0 \\ 0 & 0 & 10 \end{bmatrix}$.

$\begin{bmatrix} 2/\sqrt{5} & 0 & 1/\sqrt{5} \\ -1/\sqrt{5} & 0 & 2/\sqrt{5} \\ 0 & 1 & 0 \end{bmatrix}$

$\begin{bmatrix} 1/\sqrt{2} & 0 & 1/\sqrt{2} \\ -1/\sqrt{2} & 0 & 1/\sqrt{2} \\ 0 & 1 & 0 \end{bmatrix}$

$\begin{bmatrix} 1/\sqrt{5} & 1/\sqrt{5} & 0 \\ 2/\sqrt{5} & 2/\sqrt{5} & 0 \\ 0 & 0 & 1 \end{bmatrix}$

$\begin{bmatrix} 1/\sqrt{2} & -1/\sqrt{2} & 0 \\ 1/\sqrt{2} & 1/\sqrt{2} & 0 \\ 0 & 0 & 1 \end{bmatrix}$

$\begin{bmatrix} 2/\sqrt{5} & -1/\sqrt{5} & 0 \\ 1/\sqrt{5} & 2/\sqrt{5} & 0 \\ 0 & 0 & 1 \end{bmatrix}$

Determine a basis for the orthogonal complement, $W^{\perp}$, of $W = \text{span}\{(1,0,-1,2),(2,1,-2,5),(1,2,-1,4)\}$.

$\{(2,1,0,-1),(1,0,1,0)\}$

$\{(2,-1,0,1),(1,0,-1,0)\}$

$\{(1,0,1,0)\}$

$\{(1,0,-1,2),(0,1,0,1)\}$

$\{(1,0,-1,2)\}$

The matrix of a linear transformation $T : \mathbf{R}^2 \to \mathbf{R}^2$ with respect to the standard basis is $\begin{bmatrix} 3 & -1 \\ -1 & 2 \end{bmatrix}$. Determine the matrix of T with respect to the basis $B = \left\{ \begin{bmatrix} 2 \\ 5 \end{bmatrix}, \begin{bmatrix} 1 \\ 3 \end{bmatrix} \right\}$.

$\begin{bmatrix} -5 & -5 \\ 11 & 10 \end{bmatrix}$

$\begin{bmatrix} 5 & 0 \\ 12 & 1 \end{bmatrix}$

$\begin{bmatrix} 3 & -1 \\ -5 & 2 \end{bmatrix}$

$\begin{bmatrix} 2/5 & 1/5 \\ 1/5 & 3/5 \end{bmatrix}$

$\begin{bmatrix} 10 & -5 \\ -17 & 9 \end{bmatrix}$

If $y(t)$ satisfies the initial value problem $ty' - y = t^2$, $y(1) = 2$, find $y(2)$.

6

4

3

2

1

Find the general solution of the differential equation $y' = (1-y)^4 \cos(t)$.

$y = 1 - \big(c + 3\sin(t)\big)^{-1/3}$

$y = c - (1-y)^5 \sin(t)/5$

$y = 1 - 3\big(c + \sin(t)\big)^{1/3}$

$y = \ln|\sec(t) + \tan(t)| + (1-y)^5/5 + c$

$y = 1 - 5\big(c - \sin(t)\big)^{1/5}$

Suppose a population N satisfies the logistic equation $\frac{dN}{dt} = -\sqrt{N}(1 - N/15)(1 - N/10)$ [millions/year]. If the initial population is 12 million, determine the limiting value of the population over a long period of time.

15 million

10 million

∞

0

There is no limiting value.

The water in a 200 gallon aquarium is to be replenished by first draining out 100 gallons and then refilling the aquarium with fresh water at a rate of 2 gallons per minute. At the same time, the mixture continues to drain out of the tank at a rate of 1 gallon per minute. Determine the differential equation satisfied by $y(t)$, the amount of impurities in the tank after t minutes, while the tank is refilling.

$y' = -y/(100+t)$.

$y' = 2 - y/100$

$y' = -y/100$

$y' = t - 100/y$

$y' = 2t - 100y/(200+t)$

11. Use the Gram-Schmidt process to find an orthonormal basis for $\text{span}\{(1,0,1,0),(1,1,0,0),(0,1,1,1)\}$.

12. Consider the transformation of quadratic polynomials, $T : P_2 \to P_2$, defined by $T(p(x)) = (x+1)p'(x)$.

(a) Show T is a linear transformation.

(b) Determine the matrix of T with respect to the standard basis $B = \{1, x, x^2\}$.

13. Consider the differential equation $(6x + 2y)dx + (x + 1)dy = 0$.

(a) Show that the equation is *not* exact.

(b) Find an integrating factor, $\mu = \mu(x)$, for the equation.

(c) Use your answer in part (b) to solve the equation.